Mathematisches Forschungsinstitut
Oberwolfach

B E R I C H T

Ergodentheorie

26. bis 31.7.1965

Die in Oberwolfach vom 26.-31.7.1965 abgehaltene Tagung über Ergodentheorie stand unter der Leitung von Herrn Prof. K. Jacobs (Göttingen). Von den insgesamt 35 Teilnehmern kamen 30 aus dem Ausland.

Die Teilnehmer:

Adler, R.L.	Yorktown Heights/USA
Beck, Prof. Dr. A.	Wisconsin/USA
Benthem Jutting, L.S. van	Amsterdam
Billingsley, P.	Kopenhagen
Blum, Prof.Dr. J.R.	New Mexico
Boclé, Prof.Dr. J.	Rennes
Brown, Dr. C.C.	Göttingen
Brown, Prof.Dr. J.R.	Corvallis/USA
Brunel, Prof.Dr. A.	Sceaux/France
Cigler, Prof.Dr. J.	Groningen
Csiszár, I.	Budapest
Dowker, Frau Prof.Dr. Y.N.	London
Hajian, Prof.Dr. A.	Boston
Hanson, D.L.	Columbia/USA
Helmberg, Prof.Dr. G.	Eindhoven
Hérault, Prof.Dr. D.J.	Saint Sulpice de Favières
Hopf, Prof.Dr. E.	Bloomington/USA
Ito, Prof.Dr. Y.	Providence/USA
Jacobs, Prof.Dr. K.	Göttingen
Keane, M.S.	Göttingen
Ornstein, Prof.Dr. M.S.	Stanford/USA
Parry, Prof.Dr. W.	Brighton/England
Post, K.A.	Eindhoven

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Rényi, Prof.Dr. A.	Budapest
Révész, Prof.Dr. P.	Budapest
Roussas, Prof.Dr. G.	San José/USA
Ryll-Nardzewski, Prof.Dr.	Wroclaw/Polen
Scheffer, C.L.	Utrecht
Simons, F.I.	Amsterdam
Smit, J.C.	Nijmegen
Stone, D.M.	Rochester/USA
Sucheston, Prof.Dr. L.	Columbus/USA
Urbanik, Prof.Dr. K.	Wroclaw
Waldenfels, Dr. W. von	Jülich
Zieschang, Dr. H.	Frankfurt/M.

Leider ergaben sich einige kurzfristige Absagen; so konnten Frau A. Ionescu-Tulcea (Urbana) und die Herren J.G. Sinai (Moskau), Anosow (Moskau), Novikov (Moskau), Gottschalk Chacon (Columbus) und C. Ionescu-Tulcea (Urbana), mit deren Anwesenheit noch bis kurz vor der Tagung gerechnet worden war, nicht teilnehmen.

Die Tagung war schön und fruchtbar. Die bewährte Instituts-atmosphäre von Oberwolfach tat ihre Wirkung. Mathematiker aus den verschiedensten Ländern fanden sich zu gemeinsamer Arbeit und Diskussion und auch zu musikalischen und sportlichen Unternehmungen freundschaftlich zusammen. Insgesamt wurden 22 Vorträge gehalten und eine Reihe von Problemen gestellt. Von verschiedenen Tagungsteilnehmern wurde vorgeschlagen, der gegenwärtigen internationalen Aktivität auf dem Gebiet der Ergodentheorie durch eine jährliche Wiederholung der Tagung Rechnung zu tragen.

Nachstehend sind Auszüge der Vorträge, sowie die von den Teilnehmern aufgeworfenen Probleme zusammengestellt.

Vortragsauszüge:

R.L. Adler: Entropy of Chebyshev Polynomials.

The notion of entropy of a measure preserving transformation has a topological analogue. It can be defined for a continuous mapping of a compact topological space into itself. In this setting it is a number which is a homeomorphic conjugacy

invariant. In order to compute this number for simple examples the Chebyshev polynomials naturally arise. They are polynomials which map the interval $[-1,1]$ onto itself and as a class form a semigroup under composition. The result is that the entropy of the n^{th} Chebyshev polynomial is log n.

L.S. van Benthem Jutting: A maximal ergodic theorem for non-singular non-invertible transformations.

Dowker's proof of the maximal ergodic theorem for non-singular transformations (Jacobs, Neuere Ergebnisse der Ergodentheorie, p.64) may be applied to non-invertible transformations satisfying the condition $m(T^{-1}E) = 0 \Longrightarrow m(E) = 0$ for all $E \in B$.

P. Billingsley: Connections between ergodic theory and Hausdorff dimension.

An account of the connections between ergodic theory (the Shannon-McMillan theorem) and Hausdorff dimension, together with a review of some recent applications to dimensional problems involving continued fractions.

J.R. Blum: On roots of transformations.

Let S be the class of invertible, measurable, non-singular transformations of the unit interval onto itself. We show 1) that the transformations with roots of every order are dense in S, 2) that the ergodic transformations with no roots are dense in the antiperiodic transformations, and 3) that the antiperiodic transformations are nowhere dense in S.

J.R. Brown: Two extreme-point problems in ergodic theory.

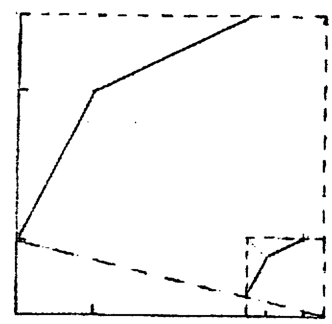
The first problem concerns the convex set M of Markov operators on $L_\infty(X,\mathcal{F},m)$ that leave the measure m invariant. Conditions are given under which operators arising from measure-preserving transformations of $(X,\mathcal{F},m)$ are dense in M, and other partial results are given concerning the extreme points of M.
The second problem concerns the extreme points of the convex set K of invariant probability measures for a fixed Markov operator T and, more generally, the minimal elements of the

cone of σ-finite, conservative, invariant measures for T. Connections with the Martin boundary are indicated.

A. Brunel: Exemple d'une transformation pour la quelle il n'existe pas de mesure invariante finie ou o-finie.

Le problème est équivalent au problème suivant: ω étant la dérivée de R.N. correspondant à T, l'équation fonctionnelle $f(Tx)\omega(x) = f(x)$ a-t-elle une solution $f \geq 0$, mesurable et finie, p.p.?

Une application T de $[0,1[$ dans $[0,1[$ est définé par son graphe (fig.). Pour tout n, on obtient des intervalles $I_{p_n},\dots,I_o,\dots,I_{p_n}$ et le tableau suivant donne les nombres d'intervalles ayant même mesure.



Nb. d'interv.	C_n^o	$2C_n^1$	...	$2^pC_n^p$...	2^n
Mesures	2^{-n}	2^{-n-1}	...	2^{-n-p}..	2^{-2n}

On définit pour toute fctn. mes., ≥ 0, h, le nombre $\alpha(h) = \int_{\{x \mid h(x) \leq 1\}} h$

Si $h_k \uparrow h$, $\alpha(h_k) \longrightarrow \alpha(h)$. Soit $E_n = \complement I_{p_n}$ et φ_n l'indicateur de E_n. G_n est l'ensemble des solutions mesurables, ≥ 0 de l'equat. approchée: $[g(Tx)\omega(x)-g(x)]\,\varphi_n(x) = 0$. Il suffit de prouver que $\alpha(G_n) = \sup_{g \in G_n} \alpha(g) \longrightarrow 0$. Ce qui se démontre en évaluant un majorant de $\alpha(g)$ d'abord pour les fonctions étagées, puis, a l'aide du lemme, pour toute autre fonctions de G_n.

J. Cigler: Ergodizitätseigenschaften von Folgen.

Für jede Folge $\omega_o = \{x_n\}$ in einem kompakten Hausdorffraum X existiert ein minimaler kompakter Hausdorffraum X_{ω_o}, eine stetige Transformation T auf X_{ω_o} und eine stetige Abbildung $\pi: X_{\omega_o} \longrightarrow X$ derart, daß $x_n = \pi(T^n\omega_o)$ ist. Dabei besteht ein enger Zusammenhang zwischen Verteilungseigenschaften der Folge $\{x_n\}$ und Eigenschaften von X_{ω_o}, T und $I_T(X_{\omega_o})$, der

Menge aller T-invarianten Maße. Es erweist sich als zweckmäßig, einen Begriff der "Pseudoergodizität" einzuführen, der für $X_{\omega_0} = X$ mit dem üblichen Begriff der Ergodizität übereinstimmt.

Dieser Begriff erlaubt u.a. eine ergodentheoretische Charakterisierung gleichmäßig gleichverteilter Folgen sowie eine einfache ergodentheoretische Deutung und Herleitung eines Gleichverteilungssatzes von van der Corput.

Für spezielle Folgen ergeben sich u.a. einfache Beweise bekannter Sätze. Beispiel: $X = T_1$ (= Gruppe der reellen Zahlen mod 1), $\omega_0 = \{p(n)\}$, $p(t) = a_0 t^k + \dots + a_k$, a_0 irrational, ergibt:

$$X_{\omega_0} = T_k,\ T(x_1,\dots,x_k) = \left(x_2,\dots,x_k, \sum_{j=1}^{k} \binom{k}{j} (-1)^{j-1} x_{k-j+1} + k!\,a_0\right),$$

$I_T(X_{\omega_0}) = m$ = Haarsches Maß auf T_k (k-dim. Torusgruppe).

Y.N. Dowker: On a mixing problem.

Let $(X,\mathcal{B},N)$ be a σ-finite measurable space with the σ-ideal of zero sets N. Let T be a 1-1 measurable, non-singular transformation of X onto X. T is called a generalized K-automorphism if there exists a subalgebra α of β such that

$$T\alpha \geq \alpha,\quad \bigvee_{i=-\infty}^{\infty} T^i\alpha = \mathcal{B},\quad \bigwedge_{i=-\infty}^{\infty} T^i\alpha = \text{trivial algebra.}$$

T is called a generalized mixing if

(1) $\nu(T^nA) - \mu(T^nA) \xrightarrow[n]{} 0$ for $A \in \mathcal{B}$, ν and μ normalized measures on $(X,\mathcal{B},N)$.

Question: If T is a generalized K-automorphism is it a generalized mixing?

Partial results: (a) If T is a generalized K-automorphism then (1) holds for every $A \in \bigcup_i T^i\alpha$

(b) If T preserves a σ-finite invariant measure m on $(X,\mathcal{B},N)$, T is conservative and α contains at least one set of finite m-measure then (1) holds also for every set A of finite m-measure.

(c) Under the same condition as in (b) $0 = \underline{\lim}\,(\nu(T^nA)-\mu(T^nA))$ for every $A \in \mathcal{B}$.

A. Hajian: Transformations of α-Type.

Let φ be an ergodic measure preserving transformation defined on a σ-finite (infinite) measure space (X,B,m). We assume the measure space is non-atomic.

Definition: φ is a transformation of type α for some α with $0 \leq \alpha \leq 1$ if $\overline{\lim_{n\to\infty}}\, m(\varphi^n A \cap A) = \alpha\, m(A)$ for every measurable set A with positive and finite measure.

We show by construction that for every α with $0 \leq \alpha \leq 1$ a transformation of type α exists which is ergodic and measure preserving with the measure of the whole space infinite.

In general it is easy to show that given any ergodic measure preserving transformation φ defined on an infinite measure space then either φ is of type 0 or of positive type (but not necessarily of a given type α). This means that either $\overline{\lim}\, m(\varphi^n A \cap A) > 0$ for all A with $m(A) > 0$ or $\lim m(\varphi^n A \cap A) = 0$ for all $m(A) < \infty$. (Details to appear).

D.L. Hanson: Roots of the one-sided N-shift.

A method for obtaining p^{th} roots of the one and two-sided N-shift with $N=k^p$ have been known for some time. A proof is given that in the case of the one-sided N-shift there is no p-th root unless $N=k^p$. The method of proof applies to other N to one transformations as well. A new method of obtaining p-th roots for the two-sided k^p-shift is given.

G. Helmberg: Über konservative Transformationen.

Ornstein und Halmos haben gezeigt, daß eine invertierbare konservative Transformation T in einem σ-endlichen Maßraum $(X,\mathcal{R},\mu)$, die $\mu(TE) \leq \mu(E)$ für alle $E \in \mathcal{R}$ erfüllt, maßtreu ist. Wird die Voraussetzung der Invertierbarkeit weggelassen, dann hat $\overset{*}{\mu}(TE) \leq \mu(E)$ für alle $E \in \mathcal{R}$ ($\overset{*}{\mu}$ = äußeres Maß induziert durch μ) noch stets $\overset{*}{\mu}(TE) = \mu(E) = \mu(T^{-1}E)$ für alle $E \in \mathcal{R}$ und "im wesentlichen" Invertierbarkeit von T zur Folge; jede der Voraussetzungen $\overset{*}{\mu}(TE) \geq \mu(E)$ für alle $E \in \mathcal{R}$ bzw. $\mu(T^{-1}E) \leq \mu(E)$ für alle $E \in \mathcal{R}$ ist mit der Maßtreue von T äquivalent, während $\mu(T^{-1}E) \geq \mu(E)$ für alle $E \in \mathcal{R}$ Maßtreue von T nicht zur Folge hat. Als Anwendung läßt sich der Zer-

legungssatz von Hopf auch auf maßtreue, nicht invertierbare Transformationen übertragen und die Maßtreue der durch eine konservative maßtreue Transformation induzierte Transformation zeigen. In den Beweisen spielen duale Aussagen zum Wiederkehrsatz für konservative Transformationen eine Rolle.

Y. Ito: Weakly wandering sequences.

Let φ be an invertible, non-singular, measurable transformation on a probability space (X,B,m).

Def. 1 $\{n_i\}$ is called a weakly wandering sequence for φ if there exists a set B of positive measure such that $m(\varphi^{n_i}B \cap \varphi^{n_j}B) = 0 \ (i \neq j)$.

Def. 2 $\{n_i\}$ is called a recurrent sequence for φ if no infinite subsequence of $\{n_i\}$ is a weakly wandering sequence. Let us now assume that φ is ergodic and admits an infinite (but σ-finite) invariant measure μ equivalent to m.

Def. 3 $\{n_i\}$ is called a dissipative sequence for φ if for every $f \in L^1(\mu)$ $\sum_{i=1}^{\infty} |f|(\varphi^{n_i}x) < \infty$ holds a.e.

Proposition 1. If $\{n_i\}$ is a weakly wandering sequence for φ, then it is a dissipative sequence for φ, and if $\{n_i\}$ is a dissipative sequence for φ, then there exists a subsequence $\{n_j\}$ which is weakly wandering. However it is not true in general that every dissipative sequence for φ is a weakly wandering sequence for φ.

Proposition 2. Let for each pos. integer k, $I_k^+ = \{ki \,|\, i=0,1,\dots\}$ and $I_k^- = \{ki \,|\, i=-1,-2,\dots\}$. Let $\{n_i\}$ be a weakly wandering sequence for φ and let $N = \{n_i - n_j \,|\, i=1,2,\dots; j=1,2,\dots\}$. Then for each k, the set theoretic differences $I_k^+ - N$ and $I_k^- - N$ must each contain infinitely many numbers.

Proposition 3. Let $\{n_i\}$ be a weakly wandering sequence for φ. Then, there exists a positive integer k such that $\{n_i\} \cup \{n_i + k\}$ is again weakly wandering for φ. (In fact, there exists infinitely many such k).

Proposition 4. Let $\{n_i\}$ be a weakly wandering sequence for φ. Then the density of n_i is zero, i.e. $\lim_{i \to \infty} \frac{i}{n_i} = 0$.

Proposition 5. φ is a positive type transformation if and only if there exists a recurrent sequence for φ.

K. Jacobs: On Poincare's recurrence theorem.

Let T be a continuous mapping of a polish space $\Omega = \{\omega, \ldots\}$ into itself. For any sequence $\mathcal{J} = t_0, t_1, \ldots$ of integers such that $0 \leq t_0 < t_1 < \ldots$ let $\Omega(\mathcal{J}) = \{\omega \in \Omega \mid \omega T^{t_k} \to \omega\}$,

$$V(\mathcal{J}) = \{m \mid m \text{ prob. meas. in } \Omega,\ mT^{t_k} \longrightarrow m \text{ (weakly)}\}$$

Put $\Omega_{rec} = \bigcup_{\mathcal{J}} \Omega(\mathcal{J})$, $V_{rec} = \bigcup_{\mathcal{J}} V(\mathcal{J})$.

Then the following results hold:

1. Recurrence theorem: If $m \in V_{rec}$ then $m(\Omega_{rec}) = 1$.
2. For any $\mathcal{J}$, $V(\mathcal{J})$ is a simplex, but not necessarily weakly compact. The extremal points of $V(\mathcal{J})$, if there are any (every mixing $m \in V(\mathcal{J})$ is extremal), are mutually orthogonal.
3. If T is 1-1 onto, with a continuous inverse, then it permutes the extremal points of $V(\mathcal{J})$.
4. If T is as in 3., and if m is an extremal point of $V(\mathcal{J})$, then either m is periodic of m is wandering:

$$mT^s \perp mT^t \qquad (0 \leq s < t)$$

In the latter case no stronger finite invariant measure exists, hence 1. is not in general a corollary of the classical Poincare's theorem.

5. Recurrent Bernoulli, Markov, and Gaussian measures can be constructed by partly intricate methods.

I have a theorem which asserts convergence a.e. of ergodic means of continuous, but not of bounded measurable functions, under assumptions on the measure which imply somewhat more than recurrence, and some independence.

W. Parry: On the coincidence of three invariant σ-algebras associated with an affine transformation.

The maximal partitions with zero entropy, quasi-discrete spectrum, and the distal property, associated with an ergodic affine transformation of a compact connected abelian group, coincide.

P. Révész: Some problems and a few results on mixing transformations.

We say that a transformation T defined on a probability space $\{X, \mathcal{S}, \mu\}$ mixes the function $f(x) \in L^2(X,\mu)$ if for any $g \in L^2$

$$E(f(T^n x)g(x)) \longrightarrow E(f(x))E(g(x)) \qquad (n \longrightarrow \infty)$$

If T mixes any element of L^2 then T is a mixing transformation. The characterization of a mixing transformation is a very complicated question (see for instance the problem of Blum in the Transaction of the New Orleans Symposium). The following type of questions will be studied:

1) Can we find an element A of $\mathcal{S}$ such that $T^{-n}A$ is a sequence of independent events if T is mixing?

2) If $\{n_i\}$ is an arbitrary sequence of integers and T mixes f, then is the strong (or the weak) law of large numbers valid for $\{f(T^{n_i}x)\}$? Conversely if the strong (or the weak) law of large numbers is valid for any n_k and any $f \in L^2$, can we state that T is mixing?

3) If we know that T mixes some $f \in L^2$, can we state that T is mixing?

C. Ryll-Nardzewski: Fixed points of semigroups of endomorphisms in linear spaces.

Let G be a certain semigroup of endomorphisms of a linear, locally convex space X and Q a G-invariant (i.e. $T(Q) \subseteq Q$ for all $T \in gG$) convex and weakly compact subset of X. G is called non-contracting on Q($G \in NC$ on Q) if for every pair x,y of different points the null vector 0 does not belong to the closure of the set $\{Tx - Ty : T \in G\}$.

Theorem: If $G \in NC$ on Q, then there exists a fixed point of G in Q. Corollary: There exists an invariant mean value of weakly almost periodic functions on arbitrary groups.

The full text will be published in the Proceedings of the Fifth Berkeley Symposium (1965).

C.L. Scheffer: On the use of martingale convergence theorems in the definition of entropy.

Using convergence theorems for martingales indexed on a directed set, it is possible to give an extension of the definition of

the Kolmogorov-Sinai entropy, which is based on the almost everywhere part of the Shannon-McMillan theorem instead of on the L_1-part of this theorem. In this way it is possible to prove for a general class of functions φ that the φ-entropy, defined below, satisfies for ergodic automorphisms

$$H^{(\varphi)} = \varphi(H)$$

where H is the Kolmogorov-Sinai entropy of the system and $H^{(\varphi)}$ is defined in the same way as H, but starting from the expressions

$$H_{\xi}^{\varphi} = \sum_{A \in \xi} \mu(A) \quad \varphi(\mu(A))$$

instead of

$$H_{\xi} = \sum_{A \in \xi} \mu(A) |\log \mu(A)|$$

This generalizes a recent result of Guseva.

F.H. Simons: Zerlegung meßbarer Transformationen in konservative und dissipative Bestandteile.

Ein weiterer Beweis eines Zerlegungssatzes für meßbare Transformationen, welchen Prof. Helmberg in der Mathematischen Zeitschrift 88 (1965) 358-367, veröffentlicht hat.

Satz: Es sei T eine meßbare Transformation in einem σ-endlichen Maßraum $(X, \mathcal{R}, \mu)$. Dann existiert eine Menge $Y \in \mathcal{R}$ mit den folgenden Eigenschaften:

1.) $T^{-1}Y \supset Y$

2.) Die Beschränkung von T auf Y ist konservativ in $(Y, \mathcal{R} \cap Y, \mu)$

3.) $Y' = \bigcup_{k=1}^{\infty} W_k$, W_k wandernde Menge.

Y ist durch 1), 2) und 3) bis auf eine Menge von μ-Maß 0 eindeutig bestimmt.

D. Stone: Ergodic orbits.

It is shown that a 1-1 ergodic measure preserving transformation T on the unit interval I is determined by the orbit $\xi_x = \{T^n x \mid n = 1,2,\ldots\}$ of almost any $x \in I$ in the following sense: A sequence $\xi = \{\xi(n) \mid n = 1,2,\ldots\}$ is a "determining sequence" (D-seq.) if it satisfies 2 (complicated) conditions. Every D-seq. ξ determines (a.e.) a unique transformation T_{ξ} by a generalization of the "continuity rule": If $\xi(n_i) \longrightarrow x$,

then $\xi(n_i+1) \longrightarrow T_\xi x$. If further ξ is uniformly distributed, T_ξ is measure preserving, but not conversely. An E-sequence is a D-seq. satisfying further (complicated) conditions. Then: Every E-seq. determines an ergodic measure preserving transformation. Conversely, if T is an ergodic measure preserving transformation, then for almost all x, ξ_x is an E-seq. and the transformation it determines is T (a.e.). Conditions are also given, in terms of the general orbit, for T to be weakly strongly mixing.

L. Sucheston: On invariant measures and maximal ergodic theorem for operators.

Theorem 1: Let K be a set of real-valued functions on X and assume that K is a linear space and a lattice under pointwise operations. Let V be a positive linear operator on K, N be a positive integer and set for an $f \in K$

$$g_N = \max_{1 \leq n \leq N} \sum_{i=0}^{n-1} V^i f, \qquad A_n = \{g_N > 0\}.$$

Then (1) $f.1_{A_N} \geq h_N - Vh_N$ with $h_N = \max(0, g_N)$ (includes Hopf's maximal ergodic theorem). Corollary. Let $K = L_\infty$, $\|V\| \leq 1$. Then for each $x \in X$, $m[U^n(f.1_{A_N})(x)] \geq 0$ where $m(x_n) = \min_L L(x_n)$.

Theorem 2: Let $T: L_1(X, \mathfrak{a}, p) \longrightarrow L_1$ be a positive linear contraction. The following are equivalent: (0) There exists an $f \in L$ with $f > 0$ and $Tf = f$. (1) $p(A) > 0$ implies $\inf_n \Pi_n(A) > 0$ with $\Pi_n(A) = \int_A T^n 1dp$.

(2) $p(A) > 0$ implies $M[\Pi_n(A)] > 0$ with $M(x_n) = \max_L L(x_n)$.

(3) All Banach limits on $\Pi_n(A)$ agree and T is conservative.

Theorem 3: T is conservative if and only if for each A, $A \subset \{\sum T^* 1_A = \infty\}$.

K. Urbanik: Entropy in Quantum Mechanics.

Two states φ and ψ of a physical system H (H - a hilbert space) are said to be equivalent with respect to the physical

quantity A (A - a self-adjoint operator in H), in symbols $\varphi \underset{A}{\sim} \psi$, if A has the same mean value at both states φ and ψ. This relation divides the set of all states into disjoint classes which are called macrostates and will be denoted by capital Greek letters Φ, Ψ, ... In the quantum macrophysics the concept of macrostates is a substitute for the concept of states (microstates). The mean value $M_A(\Phi)$ of A at the macrostate Φ is defined as the common value of mean values $(A\varphi, \varphi)$ for all $\varphi \in \Phi$. According to Jaynes principle of maximum uncertainty we define the entropy $S_A(\Phi)$ as the maximal uncertainty concerning A when the mean value $M_A(\Phi)$ is known. One can prove that $S_A(\Phi)$ is finite at every macrostate Φ if the operator A has discrete spectrum $\lambda_1, \lambda_2, \ldots$ and there exists a real number c such that $\sum_{k=1}^{\infty} e^{c\lambda_k} < \infty$. Each operator A satisfying the last conditions will be called thermodynamically regular. The motion of the system in question is determined by the Hamiltonian H. The law of the motion is customarily written as the Schrödinger equation $i\hbar \frac{\partial}{\partial t} \varphi = H\varphi$.

Theorem 1: The relation $\underset{A}{\sim}$ is invariant under the motion iff the operator A commutes with the Hamiltonian.

This theorem shows that in general macrostates with respect to operators non-commuting with the Hamiltonian branch out during the motion of the system. Therefore it is necessary to introduce a new concept of entropy at time t. We define $S_A^t(\Phi)$ to be the maximal uncertainty at time t concerning A when the initial mean value $M_A(\Phi)$ is known.

Theorem 2: (Principle of increase of entropy) $S_A^t(\Phi) \geq S_A(\Phi)$ $(t \geq 0)$.

Theorem 3: (An analogue of Boltzmann's H-theorem) Suppose that A has infinite spectrum. Then either $\overline{\lim_{t \to \infty}} S_A^t(\Phi) = \infty$ for all macrostates for which $M_A(\Phi)$ is different from the least and the greatest eigenvalue of A, or $\overline{\lim_{t \to \infty}} S_A^t(\Phi) < \infty$ for all Φ.

The first case holds whenever A and H are strongly non-commuting, i.e. spectral measures of A and H do not commute.

W. von Waldenfels: Mathematische Theorie des statistischen Starkeffekts.

Im Jahre 1913 entdeckte Stark die Verschiebung von Spektrallinien durch äußere elektrische Felder. Ist das elektrische Feld nicht zeitlich konstant und fest bestimmt, sondern ein stochastischer Prozess, so erhält man eine Linienverbreitung. Nimmt man an, das leuchtende Atom sei umgeben von einem Gas geladener Teilchen, die Teilchen seien im Raum "rein willkürlich" verteilt und besäßen Geschwindigkeiten "rein willkürlicher" Richtung aber festen Betrages, sei außerdem die mittlere Teilchendichte innerhalb des Weißkopfradius (der charakteristischen Länge des Problems) sehr klein gegen 1, so erhält man die dem Experimentator geläufigen Ergebnisse:
In der Mitte verhält sich das Linienprofil, als sei das Atom Stößen infinitesimaler Teilchen ausgesetzt. Auf den Linienflügeln hat man zu rechnen als sei das elektrische Feld wohl von Zufall abhängig, aber zeitlich konstant.

Problems.

1. Let $h_\mu(\varphi)$ denote the measure theoretic entropy of φ. Then $h_\mu(\varphi) \leq h_{top}(\varphi)$.

2. Let X be compact metric, φ homeomorphism. Then $h(\varphi) = \sup h_\mu(\varphi)$ over all regular invariant measures.

3. X compact topological group, φ continuous automorphism. Then $h(\varphi) = h_\mu(\varphi)$ where μ is Haar measure.

4. Suppose that

(1) (Ω, B) is a measurable space

(2) A is an algebra of bounded measurable functions on (Ω, B) with $1 \in A$

(3) $\varphi : \Omega \longrightarrow \Omega$ is an invertible, bimeasurable mapping

(4) X is a compact Hausdorff space

(5) for each $x \in X$, μ_x is an invariant probability measure for φ.

(6) the mapping $\mu : X \longrightarrow A^*$ given by $\mu_x(f) = \int_\Omega f d\mu_x$ is continuous for the weak * topology in A^*.

Then for each $f \in A$, $1 \leq f < \infty$

$$\lim_{n\to\infty} \int_\Omega \left| \frac{1}{n} \sum_{k=0}^{n-1} f(\varphi^k y) - \mu_x(f) \right|^p \mu_x(dy) = 0$$

uniformly for $x \in X$.

5. Does there exist a p-th root for a 2-sided N-shift such that the p-th root maps a one-dimensional set onto an ∞-dimensional set?

6. Can one exhibit a p-th root (other than the known ones) in the case of the one-sided k^p-shift?

7. In a probability space with transformation, what can be said about the rate at which $|1 - p(\bigcup_1^N T^i A)| \rightarrow 0$ under various sets of conditions?

8. Let $N(\omega, A, k) = n$ if $T^n \omega \in A$ and $T^\alpha \omega \in A$ for exactly k α's in the set $\{1, \ldots, n\}$. What can be said about the regularity of the sequence $N(\omega, A, k)$ and about its variation from the sequence $\{k$ expected first return time of a point in A to A$\}$?

9. Es sei $(X, \mathcal{R}, \mu)$ ein total σ-endlicher Maßraum, T meßbar, $M = \{W \in \mathcal{R} \mid W \wedge T^{-n}W = 0$ für alle $n \geq 1\}$. T heiße <u>rein dissipativ</u>, wenn $X = \bigcup_{k=1}^{\infty} W_k$, $W_k \in M$. T heiße <u>nicht singulär</u> falls $\mu(E) = 0 \Longleftrightarrow \mu(T^{-1}E) = 0$.

a) T sei nicht singulär und rein dissipativ. Existiert ein äquivalentes σ-endliches invariantes Maß? (Die Antwort ist ja, falls T invertierbar ist).

b) Sei T rein dissipativ und ergodisch. Muß μ rein atomar sein? (Die Antwort ist ja, falls T invertierbar ist).

c) Existiert ein Beispiel einer konservativen Transformation T (d.h. $\mu(W) = 0$ für alle $W \in M$) in einem endlichen Maßraum mit der Eigenschaft $\mu(T^{-1}E) \geq \mu(E)$ für alle $E \in \mathcal{R}$, $\mu(T^{-1}E) \neq \mu(E)$ für mindestens ein $E \in \mathcal{R}$? (Für $\mu(X) = \infty$ ist ein Beispiel bekannt).

d) Es sei X (lokal) kompakt mit abzählbarer Umgebungsbasis, $\mathcal{R}$ = Borelmengen, T stetig. Gilt $T\mathcal{R} \subset \mathcal{R}$?

10. If $X = [0,1]$ and $f(x) = x$, T mixes $f(x)$, is it true that T is mixing?

11. If X and $f(x)$ are arbitrary and the smallest σ-algebra with respect to which $f(x)$ is measurable is equal to $B(f(x)) = B$, and T mixes $f(x)$, is it true that T is mixing?

It is easy to see that if T mixes $f(x) \in L^2$ and $\{n_i\}$ is an arbitrary sequence of integers, then

$$(1) \qquad E[(\frac{1}{N}\sum_{k=1}^{N} f(T^{n_k}x))^2] \longrightarrow 0 \qquad (N \longrightarrow \infty)$$

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provided that $E(f(x)) = 0$. Conversely if for any $\{n_k\}$ (1) holds, then T mixes $f(x)$.

12. If T is mixing and $f(x) \in L^1$ is it true for any $\{n_k\}$

$$\frac{1}{N} \sum_{k=1}^{N} f(T^{n_k} x) \longrightarrow 0 \quad \text{a.e.} \quad (N \longrightarrow \infty)$$

(provided that $E(f(x)) = 0$)?

13. If T is ergodic and $\underline{\lim}\, \mu(T^{-n}A \cap A) > 0$ for any A with $\mu(A) > 0$, is it true that $\underline{\lim}\, \mu(T^{-n}A \cap B) > 0$ for any A, B with $\mu(A) > 0$, $\mu(B) > 0$?

Die Probleme stammen von R. Adler (1-3), J.R. Brown (4), Hanson (5-8), Helmberg (9) und Révész (10-13).

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