Mathematisches Forschungsinstitut
Oberwolfach

Tagungsbericht

Die Gruppen der Geometrie und die Geometrie der Gruppen

31. Juli bis 6. August 66

Wie in allen bisherigen Tagungen beschäftigte sich eine Reihe von Vorträgen (z.B. H.Bender, Ch.Hering, T.Tsuzuku) mit Kennzeichnungen geometrischer Gruppen durch Eigenschaften ihrer Permutationsdarstellungen. Weiter wurden endliche Gruppen betrachtet, die von Transvektionen bzw.Reflektionen erzeugt werden (J.E. McLaughlin, L.Solomon, J.Tits). Viele Verträge behandelten vorwiegend geometrische oder gruppentheoretische Fragen.

E.Dade und J.Tits konnten während der Tagung Probleme lösen, die sich aus anderen Vorträgen ergaben. H.Zassenhaus sandte einen Vortragsauszug, konnte aber nicht an der Tagung teilnehmen.

Teilnehmer:

Ancochea, J., Madrid
Baer, R., Frankfurt
Bender, H., Frankfurt
Böge, S., Heidelberg
Brandis, A., Tübingen
Buekenhout, F., Brüssel
Carter, R.W., Coventry
Cofman, J., London
Corbas, B., Reading
Dade, E., Pasadena
Fischer, B., Frankfurt
Green, J.A., Coventry
Heineken, H., Frankfurt
Hering, Ch., Mainz
Hughes, D., London
Huppert, B., Mainz
Johnsen, C.E., St.Barbara
Jonsson, W., Giessen
Kegel, O.H., Frankfurt
Kimberley, M., London
Livingstone, D., London
McLaughlin, J.E., Ann Arber
Salzmann, H., Frankfurt
Schellekens, G.J., Utrecht
Solomon, L., Las Crûces
Tamaschke, O., Tübingen
Tits, J., Bonn
Tsurumi, S., Tokio
Tsuzuku, T., Nagoya

Vortragsauszüge:

BENDER, H.: Eine Klasse zweifach transitiver Gruppen

Es wurde ein Beweis des folgenden Satzes skizziert:

SATZ: Sei G eine zweifach transitive Permutationsgruppe einer endlichen Menge. Der Stabilisator eines Punktes habe ungerade Ordnung und der Stabilisator zweier Punkte enthalte nur zyklische Primäruntergruppen. Dann ist G entweder auflösbar oder eine Erweiterung von $PSL_2(q)$ für eine geeignete Primzahlpotenz q.

Der Beweis beruht wesentlich auf den Arbeiten von W. Feit und N. Ito über Zassenhaus-transitive Permutationsgruppen.

BÖGE, S.: Ein Satz von Braun

Der Darstellungssatz bzw. die Maßformel für positive definite hermitesche Formen über einem imaginärquadratischen Zahlkörper (Braun, Hamburger Abhandlungen 14) sind äquivalent mit den Formeln $\tau(U) = \text{const.}$ bzw. $\tau(U) = 2$, wo τ das Tamagawamaß und U die zugehörige unitäre Gruppe bedeutet.

BRANDIS, A.: Verallgemeinerung eines Satzes von Frobenius

Sei $\mathfrak{G}$ eine endliche Gruppe, p eine Primzahl, G(p) der kleinste Normalteiler mit p-Faktorgruppe, $\mathfrak{P}$ eine p-Sylowgruppe von G, dann gilt (Wielandt):

I) $\mathfrak{P} \cap G(p) = \langle [Q, P],\ P \in \mathfrak{p} = \mathfrak{P},\ Q \in Ng(\mathfrak{p}),\ (\text{Ord}\, Q, p) = 1 \rangle$

wobei $\mathfrak{p}$ alle Untergruppen von $\mathfrak{P}$ durchläuft.
Aus I) erhält man
II) $\mathfrak{P} \cap G(p) = \langle \mathfrak{p} \cap Ng(\mathfrak{p})(p),\ \mathfrak{p} \leq \mathfrak{P} \rangle$.

BUEKENHOUT, F.: A characterization of the Miquelian inversive planes

If P is an inversive plane (Möbius-Ebene) we shall say that a collineation is an *inversion* if x, x^σ, y, y^σ are concyclic for each couple of points x, y. Each inversion is of order two. There is at most one inversion permuting points a, a' and points b, b' with a, a', b, b' concyclic,

$a \neq a'$ and $a, a' \neq b, b'$; the pair of distinct points a, a' is *admissible* if such an inversion exists for all b, b'. This leads to a classification of planes and for their collineation groups in 4 essential classes (there are 6 exceptional groups which are all finite). The most interesting result is: a plane is Miquelian if and only if each pair of distinct points is admissible.

CARTER, R.W.: Two-generator subgroups of finite soluble groups

A report was given on work due to R.W. Carter, B. Fischer and T.O. Hawkes.

Let G be a finite soluble group, $\sigma(G)$ the set of primes dividing $|G|$, $l(G)$ the nilpotent length of G and $l_p(G)$ the p-length of G. Then G contains a subgroup H generated by two elements such that $\sigma(H) = \sigma(G)$ and $l(H) = l(G)$. G also contains a subgroup K generated by two elements such that $\sigma(K) = \sigma(G)$ and $l_p(K) = l_p(G)$. Several generalisations of these theorems were also given.

COFMAN, J.: Strict semi-translation planes

Let $\mathfrak{A}$ be an affine plane of order n with a collineation group Δ possessing an orbit O of n non-collinear affine points. Let O' be the set of the intersections of the improper line with the lines of $\mathfrak{A}$ carrying at least two different points of O. If Δ is transitive on the non-degenerate triangles ABC', with $A, B \in O$, $C' \in O'$ then $n = m^2$ and $\mathfrak{A}$ contains an affine subplane a_o of order m. If Δ does not contain planar involutions, then $\mathfrak{A}$ is a strict semi-translation plane and the points of O form a desarguesian affine subplane of order m.

DADE, E.: Counterexample

Let G be a finite group, H a subgroup. We define:

1) N = the no. of ined. characters $\bar{X}$ of G such that $\bar{X}_H$ involves 1_H.
2) Two double cosets HoH, HrH in G are equivalent if

$$\frac{|HoH \cap K|}{|HoH|} = \frac{|HrH \cap K|}{|HrH|}$$

for all classes K of G.

3) M = the no. of equivalence classes of double cosets $H \sigma H$ in G.

Tamaschke conjectured:

4) N = M.

This is false. Let $G = H \cdot \underline{P}$, where H is cyclic of prime order q, $\underline{P}$ is extra-special of order p^{2a+1}, for some prime p and integer $a \geq 1$, and

5) $\underline{P}$ is normal in G.

$$[\underline{P}, H] = \underline{P} \qquad [Z, H] = \underline{1},$$

where Z is the center of $\underline{P}$. Then one gets:

6) $N \leq$ the number of classes of G $= p + \frac{p^{2a+1} - p^{a}}{pq} + p(q-1)$.

Also:

7) $H \sigma H \sim H \tau H$ if and only if $H \sigma H = H \tau H$..

Therefore:

8) M = the number of double cosets $H \sigma H = p + \frac{p^{2a+1} - p^{a}}{q}$.

Clearly the number in 6) is smaller than that in 8); for example, when $p = 3$, $a = 1$, $q = 2$, we get

$$N \leq 3 + \frac{27-3}{6} + 3 = 10 < M = 3 + \frac{27-3}{3} = 11.$$

GREEN, T.A.: Representation Algebras

If A(G) is the representation algebra of the finite group G over a field k of finite characteristic p, define for each subgroup D of G the ideal $A_D(G)$ generated by kG-modules which are D-projective. Put

$A'_D(G) = \sum_{E \subsetneq D} A_E(G)$ (sum over the proper subgroups E of D) and let

$$W_D(G) = A_D(G) / A'_D(G).$$

It is known that $W_d(G) = 0$ unless D is a p-subgroup of G, also that if D is a p-subgroup, and H a subgroup of G such that $H \supset \mathfrak{N}_G(D)$, then $W_D(H) \cong W_D(G)$.

S.B. Conlon (J.Alg. 1967) has shown that

$$A(G) \cong \Sigma\, W_D(G) \qquad \text{(isomorphism of algebras)}$$

direct sum over representatives D of all conjugate classes of p-subgroups of G. The proof rests on the lemma: for any D, the ideal $A_D(G)$ has an identity element.

HEINEKEN, H.: Commutator Properties

Let $a^{(1)} ob = ao^{(1)} b = aob = a^{-1}b^{-1}ab$, and call a set of integers M covering, if for each integer there is a multiple contained in M. The following closure properties are considered:

(CC) For each x, y of G there is an element z in the group G such that $xo(yog) = zog$ for all g in G.

(L_n) For each x in G there is an element z in G such that $x^{(n)}og = zog$ for all g in G,

(R_n) For each x in G there is an element z in G such that $go^{(n)}x = goz$ for all g in G.

Here we define inductively $a^{(n)}ob = ao(a^{(n-1)}ob)$ and $bo^{(n)}a = (bo^{(n-1)}a)oa$.

If G is a (CC)-group, G is metabelian; and if furthermore G/C(G') is noetherian, then G is nilpotent. - On the other hand, nilpotent groups of class c satisfy the condition (L_n) for all n c trivially. This set of integers n is clearly covering and the problem is, how near to nilpotence one may get by assuming (L_n) or (R_n) for all n of a covering set M. For finite groups nilpotence is obtained in the (L_n) case, while for the (R_n) case the S_3 is a counterexample. If G/G' is noetherian or artinian and G is an (L_M)-group for a covering set M, then the hypercenter of G' is contained in the hypercenter of G.

HERING, Ch.: Transitive lineare Gruppen

Eine transitive lineare Gruppe sei hier eine Gruppe von linearen Transformationen eines Vektorraumes, die auf der Menge der U verschiedenen Vektoren als Permutationsgruppe transitiv operiert. Der folgende Satz wurde diskutiert:

Sei U eine n-dimensionale transitive lineare Gruppe über GF(p) und sei nicht zugleich $n = 6$ und $p = 2$. Ist dann A ein auflösbarer Normalteiler von U und B/A ein nicht auflösbarer minimaler Normalteiler von U/A, so gilt

a) B/A ist einfach,

b) U/B ist metazyklisch und $[U:B] / n(p^n-1)$ und

c) A ist zyklisch und das Zentrum von B, ausgenommen den Fall $n = 4$, $p = 3$ und $2^5/|A|$.

HUPPERT, B.: Normalteiler und Cartergruppen

1) Sei G auflösbar, F eine gesättigte Formation und $\mathfrak{Z}$ eine deckende F-Untergruppe von $\mathfrak{G}$. Ferner sei $N(\mathfrak{G})$ der Verband der Normalteiler von $\mathfrak{G}$. Dann ist φ mit $\varphi(\mathfrak{N}) = \mathfrak{N} \cap \mathfrak{Z}$ für $\mathfrak{N} \in N(G)$ ein Verbandshomomorphismus von $N(\mathfrak{G})$ in $N(\mathfrak{Z})$.

2) Wann ist φ ein Epimorphismus?

Für spezielle Klassen von Formationen sind gleichwertig:

a) φ ist Epimorphismus.

b) Es gibt einen Normalteiler $\mathfrak{K}$ von $\mathfrak{G}$ mit $\mathfrak{G} = \mathfrak{K}\mathfrak{Z}$.

Dies gilt z.B. falls

(1) F lokal definiert durch $F(p) = F_0 \neq \emptyset$ (insbes. F: nilpotent, nilpotente Kommutatorgruppe).

(2) F: überauflösbare Gruppen.

Stimmt jedoch nicht für die arithmetisch definierte Formation, wie die der w-Hallgruppen für $w = \{p, q\}$.

JOHNSEN, E.C.: Certain Abelian Group Difference Sets

Two special classes of abelian group difference sets (AGDS's), those with the inverse multiplier and those which are skew-Hodamard, have been recently studied in Can. J. Math. 16 (1964), 787-796, and in a paper to appear in J. of Algebra. Here we answer a certain "natural" question about AGDS's and, in the process, put AGDS's into a setting whereby these two special classes become the simplest classes of AGDS's to study. We discuss some of the principal nonexistence theorems given in the above two papers.

JONSSON, W.: A Theorem of Wagner and Moufang

A projective plane π is of type Dt with respect to a non-incident point-line-pair (C,l) if for each ordered pair of distinct points (A,B) of l there is a non-trivial (A,BC)-involutory homology. With the help of a lemma of Ostrom it follows that π is (A,AC)-transitive.

A proof of the equivalence of the axiom of the fourth harmonic point and a certain doubly-restricted Desargues Theorem due to N.S. Mendelsohn was presented. By well known arguments, Moufang's Theorem on the equivalence of certain doubly restricted Desargues Theorems

and the little Desargue (provided the diagonals of no quadrangle are collinear) follows.

KEGEL, O.H.: Endliche und lokal-endliche einfache Gruppen

Ist G eine einfache lokal-endliche Gruppe, so gibt es entweder zu jeder Primzahl p eine unendliche, elementar-abelsche p-Untergruppe in G, oder aber es gibt einen kommutativen Körper K und eine natürliche Zahl n so, daß zu einer Untergruppe von GL(n, K) isomorph ist, tritt keiner der beiden Fälle ein, so sind unendlich viele einfache Gruppen Faktoren von G, die im der Liste der bekannten endlichen einfachen Gruppen von Tits, bzw. Carter nicht vorkommen.

LIVINGSTONE, D.: The doubly transitive representations of the alternating and symmetric groups

The doubly transitive representations of S_n are the canonical representations and those of degree two, and those of A_n are the canonical representations with the following exceptions:

(i) S_4 has the non-faithful representation of degree 3;

(ii) S_5 and A_5 have each one representation in S_6; associated with the outer automorphism of S_6;

(iii) S_6 and A_6 have each one representation in S_6 corresponding to the outer automorphism of S_6 and another of degree 10 associated with a maximal imprimitive subgroup of order 12;

(iv) A_7 and A_8 have each a representation of degree 15.

Note: It was pointed out by T. Tsuzuku that the question had been considered by Maillet, but details of that treatment are not at present available.

McLAUGHLIN, J.E.: Groups generated by Transvections

Let V be a vector space of dimension $n \geq 2$ over a field K. For a pair of subspaces $P \leq H$ of dimension 1 and n-1 respectively the subgroup of SL(V) generated by those transvections τ with $H = \ker(\tau - 1)$, $P = \operatorname{Im}(\tau - 1)$ is said to be of root type.

THEOREM. Take $K \neq F_2$ and let $G \leq SL(V)$ be generated by subgroups of root type. Suppose also that G is free of normal unipotent subgroups

$\neq 1$. Then for some $s \geq 1$, $V = V_o \oplus V_1 + \ldots \oplus V_s$; $G = G_1 x \ldots x G_s$; the V_j are stable for the $G_i | V_j = 1$ for $i \neq j$; $G_i | V_i = SL(V_i)$ or $Sp(V_i)$.

This answers a question raised by John Thompson.

SALZMANN, H.: Flat planes

Let $\mathbb{E} = (P, \mathfrak{L})$ be an incidence structure such that any two distinct points are joined by a unique line. Assume that P and $\mathfrak{L}$ are surfaces (2 dim. top. Manifolds), that the set of pairs of intersetting lines is open in $\mathfrak{L} \times \mathfrak{L}$ and that joining and intersection are continuous.

If P is compact, then $\mathbb{E}$ is a projective plane, its collineation-group Γ is a lie group, and $\dim \Gamma \geq 3$ iff there is a "free flag" i.e. an incident point-line pair $(p \cdot L) = F$ such that F^{Γ} is open in the flag manifold $\mathfrak{F} = (P \times \mathfrak{L} \wedge \mathfrak{S})$.

All compact flat planes with a free flag have been determined.

If P is homeomorphic to R^2, then Γ is a lie group of dimension at most 6, $\dim \Gamma \geq 3$ iff there is a free flag or if $\mathbb{E}$ is isomorphic to a parallel strip in the arguesian plane $\mathbb{D}$; $\dim \Gamma \geq 4$ iff there is a free point pair or if $\mathbb{E}$ is isomorphic to a half-plane of $\mathbb{D}$.

THEOREM: The Moulton planes are the only flat planes admitting a free point pair, in particular, these planes satisfy the parallel axiom.

SCHELLEKENS, G.J.: Generalized hexagon

Kon. Ned. Ak. Wet. A'dam A 65 = Ind. Mat. 24 (1962) 201-234.

SOLOMON, L.: Euclidean reflection groups

Let W be a finite group of linear transformations of Euclidean space which is generated by reflections. Let $\mathbb{Q}[W]$ be the group algebra of W over the rational field. We construct a decomposition of $\mathbb{Q}[W]$, relative to some system of simple roots, into 2^n left ideals, n being the rank of W. The decomposition yields a formula for the alternating character of W in terms of characters induced from parabolic subgroups.

TAMASCHKE, O.: A Generalization of Normal Subgroups

For a finite group G and a subgroup H of G the subalgebra $T_{G:H}$ of the group algebra Γ of G over $\mathbb{C}$ which is spanned by the double coset sums $\sum_{x \in HgH} x$, $g \in G$, (called the "double coset S-ring of G with respec to H") is considered as a sort of factor structure of G mod. H. This factor structure is linked with generalizations of group characters, of conjugate elements, and of normal subgroups.

1. For any representation F of $T_{G:H}$ over $\mathbb{C}$ the function

$$\varphi : g \to \varphi(g) = \frac{1}{|HgH|} \cdot \text{trace } F\Big(\sum_{x \in HgH} x\Big)$$

is called the G:H-character of F. φ is called irreducible if F is irreducible, and then it can be expressed as a sum of diagonal coefficients (considered as functions on G) of a certain irreducible representation of G.

2. $x, y \in G$ are called G:H-conjugate if $\varphi(x) = \varphi(y)$ for all irreducible G:H-characters φ.

THEOREM. x, y are G:H-conjugate if and only if $|K \cap Hx| = |K \cap Hy|$ for all conjugacy classes K of G.

3. A subgroup K of G is called H:G-normal if the subgroup average $\varphi\frac{1}{K} = \frac{1}{|K|} \sum_{x \in K} x$ is in the center of $T_{G:H}$. There exists a series of equivalent statements and properties.

THEOREM. If K is G:H-normal and $H \leq L \leq G$ then the normalizer $N_G(L)$ of L in G is contained in $N_G(KL)$. Therefore, if in addition $L \trianglelefteq G$, then $KL \trianglelefteq G$.

TITS, J.:

The following theorems, conjectured by L. Solomon, were proved during the Tagung:

THEOREM 1. Let W be a finite group generated by reflection, R a fundamental set of involutory generators of W and, for every $w \in W$, $l(w)$ the samllest length of w as a word in the elements of R. For every subset S of R, denote by Y_S the set of all $w \in W$ such that $l(rw) > \text{or} < (w)$ according as $r \in S$ or $r \in R - S$, and set $y_S = \sum_{w \in Y_S} w \in \mathbb{Z}[W]$.

TAMASCHKE, O.: A Generalization of Normal Subgroups

For a finite group G and a subgroup H of G the [illegible] of the group algebra [illegible] of G over C which is spanned by [illegible] coset sums [illegible] (called the "double [illegible]") of G [illegible] with [illegible] is considered as a sort of [illegible] structure of G mod H. [illegible] linked with [illegible] group characters, [illegible] normal subgroups.

1. [illegible]

[illegible]

is called [illegible] if [illegible] is irreducible, [illegible] as a sum of diagonal [illegible] (considered [illegible]) of a certain irreducible representation [illegible].

2. [illegible] if [illegible](x) = [illegible](y) for all irreducible [illegible].

Proof. [illegible] if and only if |K ∩ Hx| = |K ∩ Hy| [illegible] conjugacy classes [illegible].

3. A subgroup K of G is called H:G-normal if the subgroup average $\frac{1}{|K|}\sum_{x \in K}$ [illegible] center of [illegible] H:H. There exists a series of equivalent statements and properties.

THEOREM. If [illegible] is G:H-normal and H ≤ [illegible] then the normalizer $N_G(\ldots)$ [illegible] is contained in $N_G(\ldots)$. Therefore, if in addition [illegible], then HL ≤ G.

[illegible]

[illegible] theorem, conjectured by L. Solomon, was proved [illegible].

THEOREM 1. Let G be a finite group generated by reflection. [illegible] a fundamental set of involutory generators of G, [illegible] For every w ∈ G, l(w) the smallest length of w as a word in the elements of [illegible]. For every subset S [illegible] by [illegible] set of all w [illegible] such that [illegible] l(w) according [illegible] w [illegible].

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Then $\Sigma\, \mathbb{Z} y_S$ is a subring of the group-ring $\mathbb{Z}[W]$.

THEOREM 2. Let G be a group and let (B, N) be a BN-pair in G with finite Weyl group of rank 1. Let Δ be the simplicial complex associated with (B, N) ("Structures et groupes de Weyl", Sém. Bourbaki, Feb. 1965). Let m be the number of conjugates of B which are opposite to B (so that, in the case of an algebraic group over a finite field of characteristic p, m is the order of the p-Sylow subgroups). Then, $H_o(\Delta) \cong Z$, $\bar{H}_{1-1}(\Delta) = \mathbb{Z}^m$ and $H_i(\Delta) = o$ for $i \neq o, 1-1$.

TITS, J.: Algebraic groups over local fields

Report on a joined work with F. Bruhat.

TSUZUKU, T.: Some results on Permutation groups

With some other results I will talk

(1) Let G be a doubly transitive group of degree $1 + p + p^2$, where p is a prime number.
If $|G| \equiv 0\ (p^4)$, then G is alternating or symmetric.
If $|G| \equiv 0\ (p^3)$, $\not\equiv 0\ (p^4)$, then G is isomorphic to a collineation group on a projective plane over GF(p) which contains $LF_3(p)$.

(2) Let G be a doubly transitive permutation group of prime degree $p = 4q+1$, where q is also prime. If a stabilizer of one point is solvable, then

$$G \cong LF_3(3).$$

ZASSENHAUS, H.: Über die zulässigen Gitter hochsymmetrischer Bereiche

Die Geometrie der Liegruppen wird angewendet auf die Geometrie der Zahlen.

Resultate: Z.B. indefinite quadratische Formen von mehr als 4 Variablen approximieren stets Null beliebig genau mit ganzzahligen Werten der Variablen in nicht trivialer Weise. Ternäre und quaternäre indefinite Formen, die Null nicht nichttrivial approximieren, sind Vielfache rationaler Formen.