MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 10|1970

"Combinatorial Aspects of Finite Geometries"

30. März bis 4. April 1970

Die Tagung stand unter der Leitung von Prof. P. Dembowski (Tübingen und Prof. D. R. Hughes (London). In Vorträgen und Diskussionen wurden neue Ergebnisse bekannt und besprochen. Daneben konnte auch in vielen privaten Gesprächen der mathematische Kontakt vertieft werden. Die Teilnehmer kamen aus den U.S.A., Kanada, England, Norwegen, Holland, Belgien, Deutschland, Schweiz und Italien.

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Schulz, R.-H., Dr., Tübingen

Vortragsauszüge

ANDRÉ, J.: Frobeniusgroups and generalized affine spaces

A Frobeniusgroup is a permutationgroup with 1) any permutation is uniquely determined by its action on two different elements and 2) the fixpointfree permutations together with the identity form a transitive subgroup. To each Frobeniusgroup belongs an incidence structure in which two different points belong to at least one block. Replacing "at least" by "exactly" this incidence structure becomes a desarguesian affine space. It is possible to prove the structure theorems of the desarguesian affine and projective spaces in this way.

ASSMUS, E. F.: The Projective Plane of Order Ten

Let P be a projective plane of order k. Then P has no extension unless k is 2, 4, or 10. The unique planes of orders 2 and 4 do indeed have extensions.
Theorem: If there is a plane of order 10, then any two ovals meet evenly in at most 6 points. Moreover, if the plane has an extension, then there are precisely 925 ovals and any two are either disjoint or meet precisely twice.
The methods used to obtain the above result are from algebraic coding theory. They yield in fact, results about planes of order congruent to 2 modulo 4. In particular, we have an elementary proof of the well-known
Theorem: There do not exist planes of order congruent to 6 modulo 8.

(Of course, this last result is an easy consequence of the Bruck-Ryser Theorem. Our proof, however, is quite different. The above work is part of joint work with H. F. Mattson, Jr.)

BARLOTTI, A.: On anomalous spreads

Let S be a spread of a 3-space Σ. S is called an anomalous spread if there exist planes of Σ not containing any line of S. A construction is given which leads to anomalous spreads in a countable 3-space. Let G_4 be a four dimensional projective space, let Σ be a fixed

3-space of G_4 and let S be a spread of Σ. It is well known how starting from G_4, Σ and S we can get a translation plane T. (André, Bruck and Bose, Segre). If S is an anomalous spread, T will contain a subplane A such that: i) any line of T intersects in (at least) one point the subplane A, and ii) there are points of T which are incident with no line of A. So A is not a Baer subplane of T; however there are Baer subplanes of T which are isomorphis to A.

BERLEKAMP, E. R.: Combinatorial Problems in Coding Theory

One can form a sequence of extended cyclic codes of length 2^m, starting with the first order Reed-Muller code and ending with the second order Reed-Muller code, in either of two straight-forward ways. One sequence starts with the BCH codes of increasing rates; the other sequence starts with the duals of the 2-error-correcting BCH code and the 3-error-correcting BCH code. Although the answers depend on whether m is odd or even, we show that in either case the weight enumerators for all codes in both sequences are completely determined by the MacWilliams-Pless identities, the Carlitz-Uchiyama bound, the BCH bound, Kasami's weight restrictions for the second order Reed-Muller code (as discussed in Chapter 16 of Algebraic Coding Theory) and a new theorem which we recently discovered. This theorem asserts that every codeword of sufficiently low weight which occurs in the dual of any code in the sequence also occurs in the dual of the 2nd order Reed-Muller code.

VAN BUGGENHAUT, J.: harmonical lines and projective planes

We define a harmonical line as a set of (n+1) points (n odd) structured by harmonical quadruplets preserved by 8 permutations and where each product of 2 harmonical involutions has at most 2 fixed points. The group H generated by these harmonical is 2 1/2 transitive and H_{ab} has at most 2 orbits. In the case where the partial plane associated to this line is a projective plane and where the harmonical involutions are restrictions of collineations, the problem is equivalent to the following: a projective plane of order n containing an oval Q such that for each 2 distinct points of Q there exists an involutorial homology fixing these 2 points and preserving Q. The following prperties can be proved: Prop. 1: If n = 3(4) then the associated plane is Desarguesian, the oval Q is conic and H = PGL (2,n). Prop. 2: If n = 1(4) and n = $(8k-1)^2$ then the plane is Desarguesian, the oval is conic and H = PSL (2,n). In the case n = $(8k-1)^2$ it can be conjectured that the results must be the same as in Prop. 2.

COFMAN, J., DALE, C.: Automorphismgroups of finite Möbius planes

The following theorems were proved:
(a) If a finite Möbius plane of even order n admits a simple non-abelian automorphismgroup G, then either G = PSL (2,k) or G = Sz(k) (- the Suzukigroup of order $(k^2 + 1) k^2 (k + 1)$).
(b) An automorphismgroup G of a Möbius plane M acting as a primitive permutation group on the points of M is in fact 2-transitive, unless G is of odd order, in which case the number of the points of M is a prime. In both instances M is of Suzuku type or miquelian.

DEMBOWSKI, P.: Die Geometrie der Involutionen einer Gruppe

Sei G eine Gruppe, I(G) die Menge ihrer Involutionen und D(G) die Menge ihrer Dieder-Untergruppen. Interpretiert man die Elemente von I(G) als die Punkte, die von D(G) als die Geraden einer Inzidenzstruktur, so haben je zwei Punkte immer mindestens eine Verbindungsgerade, und genau dann gibt es immer nur eine Verbindungsgerade, wenn die Menge

$$P(G) = \{ \frac{1}{2} |D| : D \in D(G) \}$$

nur aus Primzahlen besteht. Eine geometrisch-kombinatorische Untersuchung der Involutionengeometrien solcher Gruppen liefert u.a. die folgenden Resultate:
Sei G eine Gruppe mit nur endlich vielen Involutionen, und es sei $2 \in P(G)$.
1. Ist P(G) = {2,n}, so n = 4.
2. Besteht P(G) nur aus Primzahlen und ist 2^e (> 2) die Ordnung einer maximalen elementar abelschen 2-Untergruppe von G, so ist entweder P(G) = {2} oder P(G) enthält einen Primteiler von $2^e - 1$.

DOYEN, J.: Some problems about Steiner systems

A Steiner triple system S(n) of order n > 7 (where n ≡ 1 or 3 mod 6) is called a non degenerated plane if every triangle generates S, a space if no triangle generates S, or a degenerated plane if S contains both types of triangles. Theorem: There exists a non degenerated plane S(n) for every n ⩾ 7 and a degenerated plane S(n) for every n ⩾ 15. The problem of existence of spaces is not yet solved.
Let N(n) denote the number of pairwise non isomorphic Steiner triple systems of order n. As a corollary to the above theorem, we have N(n) ⩾ 2 for every n ⩾ 15. Some properties of the function N(n) will be described: in particular, a stronger lower bound and an upper bound will be given.

GANTER, B.: Zur endlichen Vervollständigung endl. part. Steinerscher Systeme

Es wird der Satz bewiesen, daß sich jedes endliche partielle Steinersche System vom Typ S (2, q+1, n), wobei q die Ordnung einer affinen Ebene ist, endlich vervollständigen läßt.
Dies ist eine Verallgemeinerung eines Satzes von C. Treash.

HOHLER, P.: Eine Verallgemeinerung von orthogonalen lateinischen Quadraten auf höhere Dimensionen

Es werden Eigenschaften von "orthogonalen lateinischen Würfeln" untersucht und anschließend die Zusammenhänge von Systemen von paarweise orthogonalen lateinischen Würfeln mit den endlichen affinen Räumen diskutiert. Dabei werden an den Begriff der Orthogonalität von lateinischen Würfeln - in Analogie zum 2-dimensionalen Fall - folgende Forderungen gestellt: 1. Einfachheit der kombinatorischen Definition, 2. Ein affiner Raum soll sich als System von paarweise orthogonalen lateinischen Würfeln darstellen lassen, wobei einem Parallelebenenfeld in allgemeiner Lage ein lateinischer Würfel entsprechen soll.

HUBAUT, X.: Finite Lobatchewsky spaces

A linear space is a finite Lobatchewsky space of dimension k if
(i) there are n points on each line
(ii) the set of lines passing through a point has the linear structure of a projective space of dimension k-1

We prove the following results:

1. If k is greater than 3, there are no proper Lobatchewsky space. (a Lobatchewsky space is said to be proper if it is not an affine nor a projective space).
2. If k=3, there are only two classes of proper Lobatchewsky spaces; they are caracterized by a relation between the number of points on a line and the order of the projective space of a point. An example is provided by the S(3,6,22) which is a linear space with lines of 2 points and planes of 6 points.
3. In some particuliar cases we may prove the non existance of Lobatchewsky of dimension 3. This problem is linked to the existence of some (k,n) arcs in a projective plane.

HUGHES, D.: A characterization of free completions

The confined core of a configuration is the union of all its finite confined configurations. A configuration is openly finite (o.f.) if the number of its elements not in the core is finite.
We prove: Theorem. If a projective plane is generated by an o.f. configuration, then either the plane is equal to its confined core or it is the free completion of an o.f. configuration.

IDEN, O.: On the collineation groups of the free projective planes

a) ΓF_4 is a free product of isomorphic finite groups with a single amalgamation. Hence: ΓF_4 is hopfian and residually finite.
b) H^1_ρ in Sandler: On free extensions of rank 1 (Math.Z. 111, p. 233-248, 1969) is isomorphic with the fundamental group $\pi(\hat{\rho},L)$ at L, where $\hat{\rho}$ is the bipartite graph obtained from ρ by letting the vertex sets be the points and lines of ρ respectively and the edges be the pairs (p,1) such that p and 1 are not incident in ρ. The extension class of

$$0 \to H^{1}_{\rho} \to G^{1}_{\rho} \to \Gamma\ \pi \to 0$$

is a monomorphism.

c) Generators for $G \subset \Gamma F_n$ can be determined by the method shown in Iden: Free Planes III Math. Z. 112, p. 289-295, 1969, only if G does not contain a subgroup which fixes a nondegenerate subplane pointwise.

JOHNSEN, E.: Combinatorial Structures in Loops

E. C. Johnsen and T. F. Storer have extended the investigation of difference sets and related combinatorial structures to loops. Here, every such combinatorial structures (square tactical configuration) is identified with a corresponding structure in a loop. This involves what appears to be the first major use in the study of such combinatorial structures of the well-known theorem of König on the permutation matrix decomposition of (0,1) incidence matrices. Special forms of this decomposition are related to special algebraic properties of the loop. Two such forms are investigated and the results obtained are applied to particularly interesting classes of $\langle v,k,\lambda \rangle$ designs, namely, finite projective planes, skew-Hadamard designs, and certain self-polar $\langle v,k,\lambda \rangle$ designs. Various special constructions of difference sets and related combinatorial structures in loops and neofields are discussed.

JONSSON, W.: A Construction of the Large Mathieu Groups

With the aid of certain collections of null polarities in PG(3, 2) the steiner systems associated with the large mathieu groups can be constructed. The construction motivated a new proof of the uniqueness of these steiner systems based on properties of PG(3, 2).

VAN LINT, J.: Non-existence of perfect codes

Definitions: Let p be a prime, $q=p^{\alpha}$, $R:=(GF(q))^{n}$. Let d denote Hamming-distance, $B_{\underline{x},e} := \{\underline{y} \in R \mid d(\underline{x},\underline{y}) \le e\}$. $V \subset R$ is an e-error-correcting code if $\forall (\underline{x} \in V, \underline{y} \in V, \underline{x} \neq \underline{y})\ (B_{\underline{x},e} \cap B_{\underline{y},e} = \emptyset)$.

V is called perfect if $\bigcup_{\underline{x} \in V} B_{\underline{x},e} = R$.

Known perfect codes are: trivial codes ($n=e$ or $n=2e+1$, $q=2$), Hamming codes ($e=1$), Golay codes ($e=2$, $q=3$, $n=11$ resp. $e=3$, $q=2$, $n=23$).
Recent <u>theorem</u>: For $e=2$ or $e=3$, $q=p$, n arbitrary there are no other perfect codes (van Lint 1969).
<u>Necessary conditions:</u>

(1) $\sum_{i=0}^{e} \binom{n}{i} (q-1)^i = q^k$.

(2) The polynomial $P_e(x) := \sum_{i=0}^{e} (-1)^i \binom{n-x}{e-i}\binom{x-1}{i}(q-1)^{e-i}$ has e distinct integral zeros in $1 \leq x \leq n-1$ (Lloyd-Gleason).
We now prove: <u>THEOREM:</u> If $q=p$ a perfect e-error-correcting code of block-lenght n can only exist if (a) $q \leq e$ or (b) $p \mid e, q < 2e!+e$.
For every case not covered by the theorem we have a bound $M(e)$ such that $n < M(e)$. These remaining cases are easily checked by computer.

LIVINGSTONE, D.: <u>M_{24}</u>

It is reported that the primitive representations of 8 of the known exceptional simple groups of finite order have been classified. It is suggested that the understanding of pseudo-geometrical structures associated with the groups will be facilitated by comparison of these. In particular the Steiner systems of the Mathieu groups are greatly elucidated by their interpretation with respect to the Todd group, i.e. the imprimitive maximal subgroup of block lenght 4.

PROHASKA, O.: <u>Ableitbare Rang-3-Ebenen</u>

<u>Satz:</u> Sei P eine endliche projektive Ebene, L eine Derivationsmenge von P auf der Geraden W und A die affine Ebene mit W als uneigentlicher Geraden. Besitzt P eine Kollineationsgruppe, die L invariant läßt, auf L regulär und auf den Punkten von A als Rang-3-Permutationsgruppe operiert, so ist P eine Hallebene.

SCHRAGE, G.: Der Zusammenhang zwischen der Ordnung maximaler stabiler Punktmengen und minimaler Überdeckungen bei paaren Komplexen

Sei $X = \{x_1, \ldots, x_n\}$ und $\mathfrak{P} = P_1, \ldots, P_m$ mit $P_i \subset X$, so daß $\bigwedge_{x \in X} \bigvee_{P \in \mathfrak{P}} x \in P$ und $P_i \not\subset P_k$ für $i \neq k$. $(X, \mathfrak{P})$ heißt endlicher Komplex. In Analogie zu den in der Graphentheorie üblichen Bezeichnungen definieren wir für Komplexe die Begriffe "Weg", "Kreis", "paarer Komplex", "stabile Punktmenge" und "Überdeckung".
Satz: Ist $(X, \mathfrak{P})$ ein paarer Komplex, $M(X, \mathfrak{P})$ eine stabile Menge maximaler Ordnung und $\mathfrak{A}((X, \mathfrak{P}))$ eine Überdeckung minimaler Ordnung, so gilt $\# M((X, \mathfrak{P})) = \# \mathfrak{A}((X, \mathfrak{P}))$.
Dieser Satz liefert ein Kriterium für die maximale Ordnung stabiler Mengen für eine Klasse von Graphen, die als echte Teilmenge die paaren Graphen enthält. Als Spezialfall ist in diesem Ergebnis der Satz von KÖNIG und EGERVARY enthalten.

SCHULZ, R.-H.: Bemerkungen über die Geraden von speziellen Blockplänen

Satz: Zu jeder Auswahl von natürlichen Zahlen d, i und s mit $i \mid d$ und $d > i > s > 0$ und zu jeder Primzahlpotenz q läßt sich ein Blockplan $\mathfrak{B}$ folgender Eigenschaften konstruieren:
(a) $\mathfrak{B}$ hat die Parameter $v = q^d$, $k = q^i$ und $\lambda = 2$.
(b) Mindestens eine Gerade von $\mathfrak{B}$ hat die Mächtigkeit q^s.
(c) $\mathfrak{B}$ läßt eine transitive abelsche Translationsgruppe zu.
(d) $\mathfrak{B}$ läßt sich innerhalb der affinen Geometrie AG(d,q) darstellen.

Dieser Satz sichert u.a. (für $s \nmid i$ oder $s > \frac{i}{2}$) die Existenz von Blockplänen, die trotz einer transitiven abelschen Translationsgruppe Geraden verschiedener Mächtigkeit besitzen.
Von DEMBOWSKI und WAGNER erzielte Charakterisierungen von Blockplänen mit Hilfe der Punkteanzahl der Geraden lassen sich damit nicht analog auf Blockpläne der Eigenschaft (c) verallgemeinern.

THAS, J.: The m-dimensional projective space $P_m(M_n(q))$ over the total matrix algebra $M_n(q)$ of the n x n - matrices with elements in the Galois field GF(q) k-arcs and k-caps are defined in the m-dimensional projective space $P_m(M_n(q))$ over the total matrix algebra $M_n(q)$ of the n x n - matrices with elements in the Galois field GF(q). Particular cases are the classical k-arcs and k-caps. The following extensions of classical theorems hold:

a) The ovals of $P_2(M_n(q))$, q even, contain q^n+2 points; the ovals of $P_2(M_n(q))$, q odd, contain q^n+1 points. For every k-arc of $P_m(M_n(q))$, q even, k q^n+m+1; for every k-arc of $P_m(M_n(q))$, q odd, $k < q^n+m$.

b) A (q^n+1)-arc of $P_2(M_n(q))$, q even, is incomplete and can be completed in only one way to form a (q^n+2)-arc.

c) The ovaloids of $P_3(M_n(q))$ contain $q^{2n}+1$ points, except in the case p=q=2, n=1 where the ovaloids contain 8 points. A k-cap of $P_m(M_n(q))$, $m \geq 4$, cannot more than $q^{n(m-1)}+1$ points, except in the case p=q=1, n=1 where the ovaloids contain 2^m points.

d) Every ovaloid 0 of $P_3(M_n(q))$ (we exclude the case p=q=2, n=1) defines an inversive plane I(0); for any point P of 0 the corresponding affine plane I_P is the Desarguesian plane over the field $GF(q^n)$.

WAGNER, A.: Commuting involutions

In a projective space of odd characteristic a linear involution fixes either no points or the fixed points form 2 disjoint subspaces with span the whole space. In some cases the product of two commuting involutions of the latter type is an involution of a more "favourable" type. For example, in a five dimensional space the product of two commuting involutions, each of which fixes a pair of planes and which have a point in common, is an involution where the fixed points form a 3-dimensional space and a line. This observation gives a simple proof that a doubly transitive group on a finite 5-dimensional space of odd characteristic contains PSL. Further, using these arguments one can show that a linear collineation group of a k-dimensional finite space of odd characteristic which is transitive on $\frac{k-1}{2}$ - dimensional subspaces and contains an involution fixing some points, contains PSL.

DE WITTE, P.: An analytic method in the combinatorial theory of finite linear spaces

Let us consider a non-trivial finite linear space with p (≥ 3) points and q (≥ 3) lines. Let a denote the number of points on the line l_σ ($\sigma = 1,2,\ldots q$) and b the number of lines through the point P_α ($\alpha = 1,2,\ldots p$). We introduce the function

$$F(x) = \sum_\alpha b_\alpha^{x+1} - \sum_\sigma a_\sigma^{x+1} ,$$

and notice that F(0) = 0 trivially holds and that results of de Bruijn-Erdös-Hanani and the present author can be expressed as follows:

(a) $F(-1) \leq 0$; $F(-1) = 0$ iff the space is quasi-projective.

(b) $F(+1) \geq 0$; $F(+1) = 0$ iff the space is quasi-projective.

The new theorem is: Unless F is identically zero (in which case it is derived from a quasi-projective space), it has only the trivial root nought. The previous results, and many more, easily follow from it.

H. Kurzweil (Tübingen)