MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

T a g u n g s b e r i c h t 17/1970

MARTINGALE

17.5. bis 23.5.1970

Tagungsleiter: H. Dinges (Frankfurt/Main) und L.J. Snell (Hanover)

Während über Markoff'sche, stationäre und Gauß'sche Prozesse regelmäßig Tagungen in aller Welt stattfinden, wurde unseres Wissens den Martingalen, dem vierten Grundtyp stochastischer Prozesse, noch nie eine Spezialtagung gewidmet. Um der Gefahr auszuweichen, daß die Vorträge sich in den weitgestreuten Anwendungen der Martingaltheorie verlieren, waren die Herren D. Burkholder, J.L. Doob, P.A. Meyer und L.J. Snell für Übersichtsvorträge über die Hauptströmungen vorgesehen. Außerdem standen allen Sprechern mindestens dreiviertel Stunden für ihren Vortrag zur Verfügung: D. Burkholder, P.A.Meyer und H.Rost referierten sogar in zwei langen Sitzungen.
Es hat sich gezeigt, daß die verschiedenen Forschungsrichtungen innerhalb der Martingaltheorie einander ergänzen und in Spezialtagungen wertvolle Impulse geben können. Die Atmosphäre Oberwolfachs hat viel zum Erfolg der Tagung beigetragen. Die Teilnehmer waren sich darüber einig, daß in etwa zwei Jahren eine weitere Spezialtagung über Martingaltheorie wünschenswert wäre.

Teilnehmer:

H. Bartenschlager, Frankfurt
U. Blanke, Bochum
G.A. Brosamler, Vancouver
D. Burkholder, London
N. Dinculeanu, Bukarest
H. Dinges, Frankfurt
Catherine Doléans-Dade, Straßburg
J.L. Doob, Urbana
H. Engmann, Frankfurt
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K. Krickeberg, Heidelberg
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J.F. Mertens, Heidelberg
P.A. Meyer, Straßburg
A. Nguyen-Xuan-Loc, Aarhus
R. Peter, Frankfurt
M. Rao, Aarhus
H. Rost, Frankfurt
L. Scheller, Frankfurt
M. Silverstein, New-Brunswick
L.J. Snell, Hanover
D. Stroock, New York
S.R.S. Varadhan, New York
W. von Waldenfels, Heidelberg
J.B. Walsh, Straßburg
W. Weil, Frankfurt

Vortragsauszüge

G.A. Brosamler: Superharmonic functions: Quadratic variation and Ito formula

Let $\Omega \subset R^n$ be a Green domain, $\partial\Omega$ its Martin boundary. Let u be the difference of two positive superharmonic functions on Ω, extended to $\partial\Omega$ by the fine boundary function u*. Let Y be Brownian motion on Ω, stopped when reaching $\partial\Omega$. The process u(Y) is discussed from the point of view of quadratic variation. The Ito formula is valid for u without smothness or growth conditions.

D. Burkholder: Martingale Inequalities

Recent work on inequalities of the form

$$\mathfrak{E}\Phi(Vf) \leq c \cdot \mathfrak{E}\Phi(Uf) \quad (*)$$

is described. Here $f = (f_1, f_2, \ldots)$ is a martingale belonging to some family of martingales and U and V are operators on the family. The basic question is: When does an inequality of the form

$$\lambda^{po} \cdot P(Vf>\lambda) \leq c \cdot \|Uf\|_{po}^{po}, \lambda>0, \text{ imply } (*)\text{ ?}$$

There are two cases:

1. Φ general, f special. Here Φ satisfies $\Phi(b) = \int_0^b \varphi(\lambda)\, d\lambda$, $0 \leq b \leq \infty$ where φ is nonnegative and measurable and satisfies the growth condition $\varphi(2\lambda) \leq c \cdot \varphi(\lambda), \lambda > 0$, and f is any transform of a fixed martingale satisfying a regulary condition.
2. Φ convex, f general. Here Φ is assumed also to be convex and f is any martingale relative to a fixed sequence of σ-fields.

H. Dinges: A proof of the Martingale Convergence Theorem.

Most proofs of the convergence theorem use an estimate on upand downcrossings. It is shown that the introduction of some strictly convex function provides a natural proof: If k is positive and strictly convex and if l_y denotes the tangent in y, define $U_\delta(y) := \{x : k(x) < l_y(x) + \delta\}$ and apply the Cauchy criterion: A sequence of random variables y_n which is bounded a.e., converges a.e. iff for $\varepsilon>0$, $\delta>0$, there exists N: $P\{y_n(\omega) \notin U_\delta(y_N(\omega))$ for some $n>N\} < \varepsilon$.

J.L. Doob: <u>State Spaces for Markov Processes</u>

The general principle is: The transition function and state space of a Markov process should be chosen in such a way that the supermartingales defined in terms of the transition function are separable. This principle is illustrated in the case of Markov Chains.

W. Hansen: <u>Hunt's theorem and potential theory</u>

Let E be a locally compact space having a countable base and $\mathfrak{C}$ a family of numerical functions on E. A potential cone $\mathfrak{P}$ on E is a convex cone of real-valued continuous functions ≥ 0 on E together with an additive and positive homogeneous mapping S (called support) of $\mathfrak{P}$ into the family of compact subsets of E having a decomposition property. $\mathfrak{S}_{\mathfrak{P}}$ then is the set of all real-valued continuous functions $f \geq 0$ on E satisfying $f + p \geq q$, whenever $p, q \in \mathfrak{P}$ and $f + p \geq q$ on S(q). $\mathfrak{P}$ is said to be sub-Markov if $1 \in \mathfrak{S}_{\mathfrak{P}}$.
Using Hunt's theorem we get: $\mathfrak{C}$ is the family of excessive functions of a Feller semigroup whose potential kernel maps $\mathfrak{C}_k$ into $\mathfrak{C}_o$ if and only if $\mathfrak{C}$ is "generated" by some sub-Markov potential cone $\mathfrak{P}$ satisfying $\overline{\mathfrak{P} - \mathfrak{P}} = \mathfrak{C}_o$.

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$\mathfrak{P}$ is said to be weakly adapted, if for all $p \in \mathfrak{P}$

$$\inf \{f \in \mathfrak{F}_{\mathfrak{P}} : p - f \in \mathfrak{C}_0^+\} = 0.$$

If $\mathfrak{P}$ is weakly adapted and if for any $x \in E$ there is a $p \in \mathfrak{P}$ such that $p(x) > 0$ a thorough study of $\mathfrak{F}_{\mathfrak{P}}$ gives the existence of an $f \in \mathfrak{F}_{\mathfrak{P}}$ such that $f^{-1}\mathfrak{P} \subset \mathfrak{C}_0$. This leads to the following corollary of the preceding result:

$\mathfrak{C}$ is the family of excessive functions of a quasi-Feller semigroup if $\mathfrak{C}$ is generated by a weakly adapted sub-Markov potential cone satisfying $\mathfrak{C}_0 \subset \overline{\mathfrak{P} - \mathfrak{P}}$.

This corollary can be applied directly to get a Hunt process for any strongly harmonic space.

In Inventiones math. 5, 335 - 348 (1968) a somewhat stronger result is proved by using a generalization of Hunt's theorem.

J. B. Knight: <u>On a connection between square integrable martingales and Brownian motion</u>

Let $B_1, \ldots, B_n$ be square integrable martingales on $(\Omega, \mathfrak{A}, p)$ with natural increasing process $< B_1 >, \ldots, < B_n >$ such that $B_k^2 - < B_k >$ are martingales, $1 \leq k \leq n$. If $B_j \perp B_k$ for $j \neq k$, and if the B_k have continuous path functions, then (if $\lim_{t \to \infty} < B_k >_t = \infty, 1 \leq k \leq n$, a.s.) the processes $(B_1(\tau_1(t)), \ldots, B_n(\tau_n(t)))$, where $\tau_k(t)$ is the right-continuous inverse function of $< B_k >_t$, is an n-dimensional Brownian motion.

J.F. Mertens: <u>Repeated games with incomplete information.</u>

There is considered a special kind of games $\Gamma_n(p)$ with incomplete information, where $p = (p_1, \ldots, p_k)$ is a probability vector. It

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is shown that the values $v_n(p)$ of the games $\Gamma_n(p)$ converge to some function $v(p)$, although the game Γ_∞ (p) has in general no value. The function v is the unique solution of a system of equalities. Some differentiability properties of v and examples are given. The arguments depend heavily on martingale theory, together with the Minimax theorem.

P.A. Meyer: Survey on Stochastic Integrals

The topics considered are

1. Stochastic integrals with respect to Square integrable martingales
2. Stochastic integrals with respect to non square integrable martingales and with respect to non square integrable non-martingales.

The first talk consists entirely of classical results. In the second talk are introduced semimartingales, that is, processes X which admit a decomposition $X_t = X_o + L_t + A_t$ where L_t is a local martingale, and A_t is a process whose sample functions are of bounded variation. Stochastic integrals of predictable locally bounded processes can be defined with respect to such X. A general change of variables formula can be given as follows

$$F(X_t) = F(X_o) + \int_o^t F'(X_{s-})dX_s + \frac{1}{2} \int_o^t F''(X_{s-})d[X^cX^c]_s$$
$$+ \sum_{s\leq t} (F(X_s)-F(X_{s-})-F'(X_{s-})(X_s-X_{s-}))$$

where F is twice continuously differentiable. An application is Catherine Doléans'exponential formula: the only solution of the stochastic equation

$$Y_t = 1 + \int_o^t Y_{s-} dX_s \quad \text{is}$$

$$Y_t = \exp(X_t - \frac{1}{2}[X^c X^c]_t) \cdot \prod_{s \leq t} (1+\Delta_s X) \cdot \exp(-\Delta_s X).$$

A. Nguyen-Xuan-Loc: Strong limit of some class of projections in an L-space

Let H be a Hilbert space, $\| \; \|$ a norm on H which is weaker than the hilbertian norm and X the completion of H with respect to $\| \; \|$. For I = [o, td] let $(X_t)_{t \in I}$ be a class of subspaces of X, X_s subspace of X_t for $s \leq t$, and $P_{t,s}$ the projection $X_t \to X_s$, $t \geq s$. A martingale is defined as a collection $\{x_t : t \in I\}$ of elements of X:

1) $x_t \in X_t$ for $t \in I$

2) $P_{t,s}(x_t) = x_s$ for $t \geq s$, $s,t \in I$

Theorem: If $\{x_t : t \in I\}$ is a martingale, then the following statements are equivalent:

1. $\{x_t : t \in I\}$ is relatively compact
2. $\{x_t : t \in I\}$ is weakly relatively compact
3. $s - \lim\limits_{t \nearrow td} x_t$ exists
4. $\exists x : x_t = P_t(x)$

H. Rost: Suitable stopping of Markov processes

Let (P_t) be a (continuous or discrete) semigroup of kernels on a measurable space $(E,\mathfrak{B})$ and μ a fixed measure on $(E,\mathfrak{B})$. The problem is to characterize those measure ν which occure as stopping distributions $\nu = \mu P_T$ of the process with transition function (P_t) and starting distribution μ. The answer in the timely discrete case is given by

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<u>Theorem 1</u>: There exists a T such that $\mu P_T = \nu$ iff $\langle \mu, f \rangle \geq \langle \nu, f \rangle$ for all excessive f, i.e. for all f : $0 \leq Pf \leq f$.

There is still another theorem, analogous to Skorokhod's Lemma, in the discrete time case:

<u>Theorem 2</u>: If $\mathfrak{S}$ is a suitably defined cone of defective functions (f def.: $\Leftrightarrow 0 \leq f < Pf$) and if $\langle \mu, f \rangle \leq \langle \nu, f \rangle$ for $f \in \mathfrak{S}$, then there exists a stopping time T such that

$$\text{a)} \quad \nu \geq \mu P_T \qquad \text{b)} \quad \mathfrak{E}^{\mu}(f \circ X_T) = \uparrow \lim_n \mathfrak{E}^{\mu}(f \circ X_{T \wedge n}).$$

The method of proof consists in considering the double sequence of measures ("filling procedure"):

$$\mu_o = (\mu-\nu)^+, \nu_o = (\mu-\nu)^-, \quad \mu_{n+1} = (\mu_n P - \nu_n)^+, \quad \nu_{n+1} = (\mu_n P - \nu_n)^-,$$

$n > 0$; it shows that the desired stopping times can be defined in a "canonical" way. A slight modification of this method in case of continuous time yields the result (analogous to Th. 1)

<u>Theorem 3</u>: If (P_t) is standard and $\mu U \geq \nu U$, where μU is locally finite, then one has $\nu = \mu P_T$, where T is a suitable stopping time.

In this case T is also constructed in a canonical way.

M. Silverstein: <u>A characterization of H^p-Spaces</u>

Let D be the unit disc in the complex plane, u a harmonic function on D, $u(o) = o$; $\tilde{u}$ the conjugate harmonic to u such that $\tilde{u}(o) = o$, be $F = u + i\,\tilde{u}$. Then $u \in H^p$ means

$$\|u\|_{H^p} = \sup_{o<r<1} [(2\pi)^{-1} \int_0^{2\pi} d\vartheta \, (F(re^{i\vartheta}))^p]^{\frac{1}{p}} < \infty.$$

If T_ϑ is the convex hull of the point $\vartheta \in \delta D$ and the disc $\{z : |z| < r\}$, $o < r < 1$, define the nontangential maximal function $\mu^{**}(\vartheta) := \sup_{z \in T_\vartheta} |u(z)|$.

Let B_t Brownian motion in D, started at 0, stopped upon reaching δD and ξ the hitting time for δD. For f a function on D define $f^*(\omega) := \sup_{o<t<\xi} |f(B_t)|$.

Theorem: $u \in H^p$ if and only if $u^{**} \in L^p(d\vartheta, \delta D)$, $p > o$.

This is a new result for $o < p \leq 1$.

The proof consists of three estimates:

1. $\|u\|_{H^p} \leq \|F^*\|_p$
2. $\|F^*\|_p \leq c \cdot \|u^*\|_p$
3. $\|u^*\|_p \leq c \cdot \|u^{**}\|_p$.

L.J. Snell: Remarks on a theorem of Doob

Doob proved in 1957 that if R is an open set in N space with a Green function, u und h positive superharmonic functions on R, then $\frac{u}{h}$ has a finite fine limit at H-almost every point of $R \cup R^m$. Here R^m is the Martin boundary and H is the measure which represents h in term of maximal function. In 1963 Brelot and Doob showed how to obtain from the above theorem the fact that if u and h are strictly positive harmonic functions in the half plane then $\frac{u}{h}$ has non-tangential limits. This latter theorem is discussed from the point of view of a proof which is both relatively simple and emphasizes the role of martingale theory.

D. Strook: A.Martingale Approach to Diffusions.

Let $a(t,x) = ((a_{ij}(t,x)))_{1 \leq i,j \leq d}$, $t \geq 0$ and $x \in R^d$, be a continuous, bounded, positive definite, symmetric matrix valued function. Denote by Ω the space of continuous maps ω from $[o,\infty)$ into R^d,

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and let $x(t,\omega) = x_t(\omega)$ be the position of ω at time t. Define $\mathfrak{M}_t^s = \mathfrak{B}[x_u : s \leq u \leq t]$ and $\mathfrak{M}^s = \mathfrak{M}_\infty^s$. Given $x \in R^d$ and $s \geq 0$, it is shown that there is exactly one probability measure P on $\langle \Omega, \mathfrak{M}^s \rangle$ such that

(i) $P(x(s) = x) = 1$

(ii) for all $f \in C_0^\infty(R^d)$, $\langle f(x(t)) - \int_s^t L_u f(x(u))du, \mathfrak{M}_t^s, P \rangle$

is a martingale, where $L_u = \frac{1}{2} \cdot \Sigma a_{ij}(u,y) \frac{\delta^2}{\delta y_i \delta y_j}$.

Some consequences of this fact are discussed. In particular it is shown how this result implies the existence (and uniqueness) of a Feller process whose generator is an extention of L.

S.R.S. Varadhan: <u>Diffusion Processes and Martingales</u>

Let $G \subset R^d$ be a region such that there exists a function $\Phi(x)$ in $C_b^2(R^d)$ with $G = \{x : \Phi(x) > 0\}$, $\delta G = \{x : \Phi(x) = 0\}$ and $|\nabla\Phi| \geq 1$ on δG, $(a_{ij}(t,x))$ and $(b_j(t,x))$ diffusion coefficients in $\bar{G} \times [0,\infty)$ with

(i) $(a_{ij}(t,x))$ is continuous, bounded and uniformly positive definite,

(ii) $(b_j(t,x))$ is bounded and measurable.

Let P(t,x) be a bounded, continuous, nonnegative function on $\delta G \times [o,\infty)$ and $\gamma(t,x)$ an R^d-valued function on $\delta G \times [o,\infty)$ which is bounded, continuous and satisfies the inequality $\langle \gamma(t,x), (\nabla\Phi(x) \rangle \geq \beta > 0$ on $\delta G \times [o,\infty)$. The problem is to find a process corresponding to the diffusion coefficients a,b in G with the boundary condition $Pu_t + \langle \gamma, \nabla u \rangle = o$ on δG:

Let $s \geq o$ and $x \in G$. P is called a solution to the submartingale problem if P is a measure on $C[[s,\infty)\times \bar{G}]$ such that

(i) $P\{\omega : x(s,\omega) = x\} = 1$ and

(ii) For every $u \in C_b^{1,2}[[s,\infty)\times \bar{G}]$ with $Pu_t + \langle \gamma,\nabla u \rangle \geq 0$ on $[s,\infty)\times \bar{G}$,

$$u(t,x(t)) - \int_s^t (u_\lambda + \tfrac{1}{2}\Sigma a_{ij}\frac{\delta^2 u}{\delta x_i \delta x_j} + \Sigma b_j \frac{\delta u}{\delta x_j})(\lambda,x(\lambda))\cdot \chi_G(x(\lambda))\, d\lambda$$

is a submartingale.

Theorem 1: For each s, x at least one solution exists.

Theorem 2: Under the additional assumptions

(i) γ satisfies a Lipschitz condition in t and x

(ii) P is either identically zero or is strictly positive and satisfies a Lipschitz condition (can be dropped in the homogeneous case)

for each s,x the solution $P = P_{s,x}$ is unique.

Corollary: $\{P_{s,x}\}$ defines a strong Markov Feller process.

John B. Walsh: Birth of a Process

There are considered two examples ob subprocesses of a given strongly Markov process X:

A random variable L is an L-time if 1) $L \leq \xi$ (= lifetime)
2) $L\circ\vartheta_t = (L-t)^+$. Define "killing operators k_t" by

$$(X\circ k_t)_s = \begin{cases} X_s & s<t \\ \Delta & s\geq t \end{cases} .$$

An L-time is a co-terminal time if in addition to 1) and 2):
3) $s > L \Rightarrow L\circ k_s = L$.

Theorem 1: Let L be an L-time. The process $Y_t = \begin{cases} X_t & t<L \\ \Delta & t\geq L \end{cases}$

is again strongly Markov with semigroup

$$P_t^{\Phi}(x,dy) = \frac{\Phi(y)}{\Phi(x)} P_t(x,dy)$$
$$0 \text{ if } \Phi(x) = 0$$

where $\Phi(x) = P^x\{L>0\}$.

Given a coterminal time L, there is an associated terminal time T_L:

$$T_L = \begin{cases} \inf\{t : L \circ k > 0\} \\ \infty \text{ if above is empty} \end{cases} = \text{"}1^{st} \text{ time L become interesting"}:$$

If Q_t is the semigroup of the process killed at T_L and $\Psi(x) = P^x\{L=0\}$ we have a martingale.

Theorem 2: If L is an exact terminal time, the process $(Z_t, t > 0)$ where $Z_t = X_{L+t}$ is strongly Markov with transition probabilities Q_t^{Ψ} : $(Q_t^{\Psi}(x,dy) = \frac{\Psi(y)}{\Psi(x)} Q_t(x,dy))$.

H. Bartenschlager