MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

T a g u n g s b e r i c h t 25/1970

Nonstandard-Analysis

19.7. bis 25.7.1970

Zum ersten Male fand in Oberwolfach eine Tagung über das junge Gebiet der Nonstandard-Analysis statt. Tagungsleiter waren D.Laugwitz (Darmstadt) und W.A.J. Luxemburg (Pasadena). Außer den Vorträgen fanden zahlreiche Seminare und Diskussionen über relevante Themen statt, z.B. über Publikationen und Vereinheitlichung der Terminologie.

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A.ROBINSON: Nichtarchimedische Körper

Vor zwei Jahren hat D. Laugwitz den Körper L der verallgemeinerten Potenzreihen $\Sigma a_k t^{\nu_k}$ (a_k, ν_k reell, $\nu_o < \nu_1 < \nu_2 < \to \infty$) untersucht, der schon früher von Levi-Civitá und Ostrowski betrachtet wurde. Er hat gewöhnliche, unendlich differenzierbare Funktionen auf diesen Körper erweitert und die Frage aufgeworfen, ob diese den Zwischenwertsatz und den Mittelwertsatz erfüllen. Hier wird gezeigt, daß dies nicht immer der Fall ist, wohl aber unter ausdrücklich formulierten sehr allgemeinen Bedingungen. Diese Ergebnisse werden erhalten durch die Einbettung von L in einen Restklassenkörper eines Unterringes eines nichtstandard Modells (genauer, z.B. eines Ultraprodukts) $^*R$ der reellen Zahlen.

W.A.J. LUXEMBURG: Some new applications of nonstandard analysis

P. Rosenthal (sec. Math. Monthly 76 (1969) p.925) asked the following question: Are almost commuting matrices near commuting matrices! It was shown by means of nonstandard analysis a positive answer of the following form can be given. Let $M(C^n)$ denote the set of all complex nxn-matrices with norm $||A||=\Sigma|a_{ij}|^2$ and let $M_1(C^n)$ be the unit ball (n = 1,2,...). Then we have the following result. For every n = 1,2,... there exists a mapping $f_n(\varepsilon)$ of $R^+ = \{x: x > o\}$ into R^+ with the following properties: (i) $f_n(\varepsilon) \to o$ as $0 < \varepsilon \to 0$, (ii). For all $A,B \in M_1(C^n)$ such that $||AB-BA|| \leq \varepsilon$ there exists matures $A',B' \in M_1(C^n)$ such that

$A'B' = B'A'$ and $||A - A'|| \leq f_n(\varepsilon)$, $||B - B'|| \leq f_n(\varepsilon)$.

For details we refer to the paper: W.A.J. Luxemburg & R.F. Taylor, Almost commuting matrices are near commuting matrices. Kon. Nederl. Akad. Wetensch, Proccedings A 73 = (Indag. Math. 32) 96-98 (1970) .

L.D.KUGLER: Almost Periodicity on Groups and Semigroups

Von Neumann's theory of almost periodic on an arbitrary group G, and generalizations thereof, can be approached from the point of view of non-standard analysis. Among the applications of this method is an intuitively appealing construction of group compactifications on which the continuous functions constitute an isometric, isomorphic image of almost periodic functions on G. The work of DeLeeuw and Glicksburg (Acta Math. 105, 1961), in which the more general weakly almost periodic functions are similarly analyzed, may also be considered from the nonstandard point of view.

The underlying idea of the nonstandard approach to almost periodic compactifications is to consider an enlargement $^*G$ of G and study the set of so-called "near-periods" of an almost periodic function f, namely, those elements t of $^*G$ for which f(xty) and f(xy) differ by an infinitesimal for all x and y in $^*G$. The compactifications are obtained by simply taking quotient structures.

Ch.PURITZ: Choquet's Ultrafilters and Skies in $^*N$

Two numbers $a,b \in {}^*N-N$ are in the same sky if there exist standardfunctions f,g s.t. $f(a) \geq b$ and $g(b) \geq a$. If equality holds, a and b are in the same constellation. A sky is linked if it consists of one constellation. If $^*N$ is N^N/U the skies are related to properties of U studied by G. Choquet, in a different context. Assuming Continuum Hypothesis (CH) $^*N-N$ can be a single sky, even a single constellation, or can consist of 2,3,4 ... n ... or of c skies, which can be linked or unlinked. In an enlargement there is no highest sky and the skies are ultimately densely ordered. (If K-saturated, K>c, ordering is dense all the way.)

A free filter on N or R has non-empty monad in every proper extension iff (CH) it has an elementary refinement, and in an enlargement the monad of a free filter $\overset{J}{\wedge}$ on N meets every sky at most once iff J is rare (in Choquet's sense).

If (X,p) is a metric space, a point $a \in {}^*X$ is serial if it belongs to ${}^*S$ for some countable $S \subset X$.

We obtain an expression for the monad of a in the p-topology, which involves skies.

G.JANSSEN: Eingeschränkte Ultraprodukte von W^*-Algebren vom Typ II_1

Das Ultraprodukt der metrischen Räume $(M_i, d_i), i \in J$ bezüglich eines nichttrivialen Ultrafilters U in J ist ein Raum M^* mit einer "Metrik" d^*, deren Werte in der entsprechenden Ultrapotenz $\mathbb{R}^*$ von $\mathbb{R}$ liegen. Nach Wahl von $e^* \in M^*$ werden die Nicht-Standard-Anteile mittels einer bereits von Artin-Schreier in "Algebraische Konstruktion reeller Körper", Hamb. Abh. 5 (1926) verwendeten Idee wieder entfernt, und man erhält als eingeschränktes Ultraprodukt einen gewöhnlichen metrischen Raum $(M(e^*), d)$.

Lemma: Mit (M_i, d_i) ist auch $(M(e^*), d)$ vollständig.

Im Fall metrischer Vektorräume ist das eingeschränkte Ultraprodukt eindeutig durch die Wahl $e^* = 0 \in M^*$. Folgende Eigenschaften bleiben erhalten: Metrischer Vektorraum, normierter Vektorraum, Banachraum, normierte Algebra, Banachalgebra, W^*-Algebra vom Typ II_1 etc.

M.WOLFF: Completion of Cauchy-Algebras

We consider Ω-Algebras in the sense of Universal Algebra, where Ω consists only of operations besides the equality relation and some relations on Ω. Given an Ω-Algebra with supporting set $A \neq \emptyset$ (by definition is $A \neq \emptyset$) we select a certain subset γ of the set of all filters $f \neq \mathcal{P}(A)$ (=:0) and call the elements of γ Cauchy-filters and $(\mathfrak{A},\gamma)$ a Cauchy-Algebra if the following conditions are satisfied:

(i) $a \in A \Rightarrow \dot{a} \in \gamma$ (where $\dot{a}$ is the filter generated by $\{a\}$)

(ii) $f \in \gamma$ and $\mathcal{G} \leq f$ (that means $\mathcal{G} \supset f$ in set theoretical terms) $\Rightarrow \mathcal{G} \in \gamma$

(iii) If $\mathcal{G}_1, \mathcal{G}_2 \in \gamma$ and $\mathcal{G}_1 \wedge \mathcal{G}_2 \neq 0$ (i.e. the filter generated by $\{G_1 \cap G_2 : G_i \in G_i\}$) then $\mathcal{G}_1 \vee \mathcal{G}_2 \in \gamma$ ($\mathcal{G}_1 \vee \mathcal{G}_2$ is the filter $\{G_1 \cup G_2 : G_i \in \mathcal{G}_i\}$)

(iv) γ is an Ω-Algebra, where the n-ary Operation ω is definid by $\omega(f_1,\dots,f_n) = \omega(f_1 x \dots x f_n)$ (i.e. the filter generated by the sets $\omega(F_1 x \dots x F_n)$ with $F_i \in f_i$)

$(\mathfrak{A},\gamma)$ is called separated iff $\dot{a} \vee \dot{b} \in \gamma$ implies always $a=b$.

$(\mathfrak{A},\gamma)$ is called complete iff for all $f \in \gamma$ there exists an element $b \in A$ with $f \vee \dot{b} \in \gamma$.

If $f \vee \dot{b} \in \gamma$ we say: f converges to b and get the associated limes-space. Let now $(\mathfrak{A},\gamma)$ be fixed and look at an enlargement $^*M$ of the whole structure over A. $^*A$ is in a natural way considered as an Ω-algebra and $S = \bigcup \mu(f)$, where $\mu(f)$ denotes the filter-monade of $f, f \in \gamma$ is a subalgebra. Under slight restrictions on $(\mathfrak{A},\gamma)$ $R = \{(a,b) \in S x S : \exists f \in \gamma \text{ with } a,b \in \mu(f)\}$ is a congruence relation on S. We consider now the Ω-algebra $A_S = S/R \overset{q}{\leftarrow} S$ where q denotes the canonical mapping.

Let $\mathfrak{F}_s$ = {f'filter on A_s : $\exists f \in \mathfrak{F}$ with $f' \leq f^q$}where f^q is generated by {q(*F ∩ S) : F ∈ f standard}. Under a few more assumtions ($\mathfrak{A}_s$, $\mathfrak{F}_s$) turns out to be an Ω-algebra which is separated and complete. An Ω-morphism T : ($\mathfrak{A}$, $\mathfrak{F}$) → ($\mathfrak{A}'$, $\mathfrak{F}'$) is uniformly continuons iff T maps $\mathfrak{F}$ into $\mathfrak{F}'$. The canonical mapping a → q (*a) maps a uniformly continuons into ($\mathfrak{A}_s$, $\mathfrak{F}_s$) the so-called nonstandard completion of ($\mathfrak{A}$, $\mathfrak{F}$) and ($\mathfrak{A}_s$, $\mathfrak{F}_s$) turns out to be an universal solution for the embedding functor of a subcategory of separated complete Ω-algebras. Here once more we have to consider new requirements, which all are necessary and sufficient for solving the problem of reflectivity of the specified subcategory. The method applies to vector lattices over R with star-convergence. And the theory includes the usual one of A. Weil uniform spaces.

J.HIRSCHFELD: Ultrafilters and ultrapowers in Non Standard Analysis

Let M be the full model based on a set U (with all possible relations). Let I be a set of indices. We embed all ultrapowers of the form M^I/F into an enlargement of M and use it to show that every elementary embedding of one ultrapower into another is induced by some function from I to I .

We give model theoretic properties of M^I/F (where M and I are countable) which are equivalent to saying that F is an absolute filter or a p-filter. Finally we prove some properties of extensions of M which do not contain submodels of the form M^I/F, where F is p or absolute.

L.YOUNG: Non-Standard Analysis and Topological vector spaces

The basic concepts and results of the theory of topological vector spaces are treated with nonstandard methods, the concept of topological samifield being used as an expository tool: Characterisation of equicontinuity using the polar of the monad of 0 and related results; the theorems of SMULIAN, BOURBAKI-ALAOGLU and MACKEY-ARENS. We derive the following characterisation of pre-near-standard points:

$x \quad \hat{E}$ has $O(E,F)$ finite seminorm; C a saturated cover of F by $O(F,E)$ bounded closed convex circled sets. The following are equivalent:

1. x is $\emptyset_e$ pre-near-standard.
2. $\emptyset_e\,(x-\tilde{x}) \simeq 0$ and $\tilde{x}$ is $o(F,E)$ continuons on members of C.
3. $y \quad \nu(C) \quad \mu_F\,(0) = \langle xy \rangle \simeq o$
4. $x \quad \nu(e) \quad {}_F\,(o) \quad {}^{o}$

Here $\tilde{x}$ is the functional on F defined by $\tilde{x}(y) = {}^{o}|\langle xy \rangle|$; $\mu_F\,(o)$ the weak monad of 0; $\emptyset_e$ the seminorm over the semifield R^C corresponding to the topology of uniform convergence on members of C; $\nu(C) = \quad \{\hat{S}|S \quad C\}$.

There are ten corollaries which include almost all known results on precompactness and the completeness theorems of GROTHENDIECK.

A.JENSEN: A computer oriented version of non-standard analysis

Non-standard numbers may be introduced in programming languages by allowing of symbols (infinites) $N_1, N_2, \ldots$ defined by the property that expressions $U(N_1, N_2, \ldots, N_n)$ should be calculated as $\lim_{N_1} \lim_{N_2} \ldots \lim_{N_n} U(N_1, N_2, \ldots N_n)$.

If $\lim_n U(n)$ is definid to have value $\alpha \pm \mu$ (α and $\mu > o$ are rationals) provided $\{i \in N \mid |u(i) - \alpha| < \mu\}$ lies in some prechosen ultrafilter on N then one obtains a non-standard theory with a hierarchy of equivalence relations, which allows a smooth unified treatment of standard and internal functions, operators, etc., and which is well suited for elementary calculus teaching because computation of reasonable expressions involving infinites can actually be simulated on electronic computers.

I.JUHÁSZ: Non-standard notes on the hyper space

The hyper space H(R) of a space R consists of all closed subsets of R, and its topology is determined by the (open) base

$$\mathcal{L} = \{\langle O_1, \ldots, O_n \rangle : O_i \text{ is open in } R\},$$

where a closed A R belongs to $O_1, \ldots O_n$ iff

(i) $A \subset \bigcup_{i=1}^{n} O_i$ and (ii) $A \cap O_i \neq \emptyset$ for each $i = 1, \ldots, n$.

In this note we shall give simple non-standard characterisations of the topology of the hyper space, i.e. characterise the monads of points of the hyper space. Using this characterisation we give simple proofs o f the following known results:

(1) If R is compact, so is H(R).

(2) If R is T_1 and normal, then H(R) is completely regular.

M.MACHOVER: Infinitesimal paths

Consider a point p in a topological space X. X is pathwise locally connected at p iff for every x $\mu(p)$ there is a path entirely contained in $\mu(p)$, and leading from p to x. In a sufficiently saturated enlargement, we define a homotopy relation between closed paths contained in $\mu(p)$ and based on p. The homotopy classes form a group G_p with respect to composition of paths modulo homotopy. G_p is trivial iff X is semi-locally pathwise simply connected at p . G_p can be used to investigate the local connectivity properties of X at p.

P.A.LOEB: A non-standard representation $\mathcal{L}^\infty$

Let (X,M) be a measurable space with a family of "null sets" N⊂M. Let $\|\ \|_\infty$ and $\mathcal{L}^\infty$ be the usual sup-norm and Banach space. Let $P = \{A_i \in {}^*M : 1 \leq i \leq \omega_p\}$ be a finite partition of X so that $X = \bigcup_1^{\omega p} A_i$, $A_i \cap A_j = \emptyset$ if $i \neq j$ and $\forall B \in M$, ${}^*B$ is exactly the union of an internal subcollection of P. If $\|f\|_\infty < \infty$ and $A_i \in P$, $\sup_{A_i} {}^*f - \inf_{A_i} {}^*f \simeq 0$. Let $O_n = \bigcup \{A_i \in P:$

$A_i \in {}^*N\}$. $O_N \in {}^*N$, and $B \in N \Rightarrow B \subset O_N$. Let C_p be a choice function for P. For measurable f with $\|f\|_\infty < \infty$, set $T(f)(i) = f(C_p(i))$ if $A_i \notin {}^*N$, and set $T(f)(i) = 0$ if $A_i \in N$. Then $T(f) = T(g)$ when $\|f - g\| = o$, so T given a 1 - 1 linear mapping of $\mathcal{L}^\infty$ into R^ω . Also, $\max_i |T(f)|(i) \simeq \|f\|_\infty$.

The mapping T can be extended to the set Φ of all finitely additive, finite measure by setting $T(\mu)(i) = \mu(A_i)$ for each $i, 1 \leq i \leq \omega_p$. On the other hand, if $e \in {}^*R^\omega$, and Σe_i^t and Σe_i^- are finite, then setting $\emptyset(e)(B) = \sum_{A_i \subset B} e_i$ for each $B \in M$, we have $\emptyset(e) \in \Phi$. If $[f] \in \mathcal{L}^\infty$ and $\mu \in \Phi$, then

$$\int_B f\, d\mu = \sum_{A_i \subset B} T(f)(i) \cdot T(\mu)(i)$$

for $B \in M$. Each bounded linear functional F on $\mathcal{L}^\infty$ is given by $\emptyset(\{F(X_{A_i})\})$.

D.LAUGWITZ: Nichtarchimedische Körper für die Messung von Berührungswinkeln

Bei Euklid gibt es sowohl in den Definitionen als auch in einigen Sätzen Winkel zwischen krummlinigen Schenkeln, die von geradlinigen Winkeln deutlich unterschieden werden. Diese Winkel, welche Berührungswinkel heißen, wenn die Schenkel einander berühren, spielten lange Zeit eine große Rolle.

Es mögen zwei von einem Punkt P_o der Ebene ausgehende Kurven äquivalent heißen, wenn sie in einer Umgebung von P_o zusammenfallen. Die Klassen heißen Kurvenkeime. Geordnete Paare von Kurvenkeimen heißen Winkel. Ein Winkelmaß ist eine Abbildung der Winkel in eine additive Gruppe. Es werden verschiedene Winkelmaße diskutiert, ihre nichtarchimedische Anordnung wird untersucht, und die Isomorphie verschiedener Winkelmaße bewiesen.

K.D.STROYAN: Additional Remarke on the Theory of Monads: Compactifications and Monadie Closure Operators

The idea of neighborhood monad in a *-topological space of Robinson and Luxemburg as well on the discrete monads of Luxemburg can be unified to monads with respect to a ring of sets (or distributive lattice with 0 & 1). These monads can then be used to describe the Wallman compactifications. (The Cech-Stone case was investigated by Luxemburg u. Robinson.) The topology is described an X by a monadic closure operation (i.e. all closed sets are monads of a certain kind.)

The standard topology of X can be recovered from X by using μ (the neighborhood monads) as a closure operation on near standard points.

All (Hausdorff) compactifications arise via monadic closure operations as can be seen by nonstandardization of the Samuel compactification.

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