MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

T a g u n g s b e r i c h t 19/1972

Gruppentheorie

7.5. bis 13.5.1972

Die diesjährige Gruppentheorietagung wurde von Prof.W.Gaschütz (Kiel) und Prof.K.W.Gruenberg (London) geleitet. Hauptsächlich wurden Fragen aus der Darstellungstheorie und der Theorie der linearen Gruppen, aber auch andere Themen über endliche und unendliche Gruppen behandelt, so daß sich ausreichend Stoff für vertiefende und anregende Diskussionen ergab.

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Vortragsauszüge

A. DRESS: Schwach äquivalente Darstellungen von Gruppen in Gittern

Mit Hilfe multiplikativer Induktionstechniken wird der folgende Satz:

" G endliche Gruppe, R kommutativer Ring mit 1, $\frac{1}{p} \in R$, Γ, Γ' RG-Gitter, $\Gamma|_H$ stabil isomorph zu $\Gamma'|_H$ für alle $H \leq G$ mit zyklischer p-Sylowgruppe $H_p \Longrightarrow \Gamma$ stabil isomorph zu Γ' " auf die Untersuchung elementar abelscher Gruppen der Ordnung p^2 zurückgeführt und dort leicht verifiziert.

VERENA HUBER DYSON: Zum Entscheidungsproblem der Theorie endlicher Gruppen

Es wird ein System von endlich erzeugten Gruppen L(S) konstruiert, wobei S die Mengen von ganz rationalen Zahlen durchläuft. L(S) ist genau dann residual endlich, wenn das Komplement von S eine Vereinigung von arithmetischen Progressionen ist. L(S) ist genau dann endlich (rekursiv) präsentierbar, wenn S endlich (rekursiv aufzählbar) ist, und L(S) hat genau dann ein lösbares Wortproblem, wenn S rekursiv ist.

Passende Wahl von S liefert Gruppen, die

(1) residual endlich und rekursiv präsentierbar sind, aber ein unentscheidbares Wortproblem haben und sich infolgedessen nicht in endlich präsentierbare residual endliche Gruppen einbetten lassen,

(2) ein entscheidbares Wortproblem haben, während das Wortproblem für ihre maximalen residual endlichen Faktorgruppen unlösbar ist.

Die Bedeutung dieser Resultate für das Entscheidungsproblem der universellen Theorie endlicher Gruppen wird analysiert.

W. GASCHÜTZ: Zur exakten Kohomologiesequenz für Gruppen von Hochschildt-Serre

Für die Exaktheit der genannten Sequenz wurde ein Beweis angegeben, der insofern elementar ist, als er die Theorie der spektralen Sequenzen vermeidet und sich nur auf die Eigenschaften der gewöhnlichen exakten Kohomologiesequenz mit dem Verbindungshomomorphismus stützt.

G. GLAUBERMAN: Direct factors in Sylow 2-subgroups

For every finite group G let $Z^*(G)$ be the subgroup of G that contains $O_{2'}(G)$ and satisfies the condition that $Z^*(G)/O_{2'}(G) = Z(G/O_{2'}(G))$.

Theorem: Let Q be a finite non-Abelian 2-group. Assume that $\Omega_1(Q) \subseteq Z(Q)$ and that Q is not decomposable as a direct product of non-identity subgroups. Then the following are equivalent:

(a) For every subgroup A of the automorphism group of Q of odd order, A fixes some non-identity element of Q.

(b) For every 2-group R and every finite group G that contains Q R as a Sylow 2-subgroup, $Z^*(G) \neq 1$.

Corollary: Let G be a finite group. If G has a Sylow 2-subgroup of the form $Q \times R$ for some generalized quaternion group Q, then $Z^*(G) \neq 1$.

K.W. GRUENBERG: The decomposability of relation modules of finite groups

Let G be a finite group, $1 \longrightarrow R \longrightarrow F \xrightarrow{\pi} G \longrightarrow 1$ a free presentation with F finitely generated. The decomposability of the corresponding relation module $\bar{R} = R/R'$ is discussed.

Let ϑ be the class of all G for which a minimal relation module is decomposable.

Sample of results: (1) $G \notin \vartheta$ if G is soluble, or G has ≤ 2 generators ($d(G) \leq 2$). (2) Suppose $N \trianglelefteq G$ so that $N = N'$, $d(N) < d(G)$ and $d(G/N) < d(G)$. Then $G \in \vartheta$. (E.g. $G = A_5^{20}$). (3) Suppose $G \in \vartheta$ and A is a minimal normal subgroup and A is abelian. Then $d(G) = d(G/A)$ and $G/A \in \vartheta$.

K.W. GRUENBERG: Frattini extensions

Let K be a commutative ring, G a group and M an irreducible KG-module. Let ϑ be the set of isomorphism classes of extensions $1 \longrightarrow M^{(s)} \longrightarrow E \longrightarrow G \longrightarrow 1$ that have the property of being essential: if $E = HM^{(s)}$, $H \cap M^{(s)}$ a K-submodule, then $E = H$. Then ϑ has the structure of a projective geometry over $D = End_{KG}(M)$ provided $\dim_D H^2(G,M)$ in finite.

H. HEINEKEN: Torsionsfreie lokal auflösbare Gruppen mit Min - n.

Es gibt überabzählbar viele torsionsfreie lokal auflösbare Gruppen, die die Minimalbedingung für Normalteiler erfüllen, und paarweise nicht isomorph sind. Genauere Beschreibung der von J.S. Wilson und H.Heineken konstruierten Gruppen dieser Art.

H. KRÄMER: Über die Automorphismengruppen des Burnsideringes endlicher abelscher Gruppen

Bezeichne für eine endliche Gruppe $\Omega(G)$ den Burnsidering von G, Aut$\Omega(G)$ die Automorphismengruppe von $\Omega(G)$.

Satz 1. Sei G nilpotent, G_p = p-Sylowgruppe von G. Dann gilt

$$\mathrm{Aut}\,\Omega(G) = \prod_{p \mid |G|} \mathrm{Aut}\,\Omega(G_p).$$

Sei nun G abelsch, o.B.d.A. p-Gruppe. Man macht Aut$\mathcal{V}(G)$, $\mathcal{V}(G)$ = Untergruppenverband von G, zu einer Untergruppe von G.

Satz 2. Sei G abelsche p-Gruppe.

Ist G zyklisch, dann gilt Aut$\Omega(G) \cong Z/2Z$.

Sei nun G nicht zyklisch. Ist $p \neq 2$, dann gilt

$$\mathrm{Aut}\,\Omega(G) = \mathrm{Aut}\,\mathcal{V}(G).$$

Ist p = 2 und $G: F(G) \geq 8$ (F(G) = Frattiniuntergruppe von G) oder $G: F(G) = 4$ und G vom Typ $(2^m, 2^n)$; $m,n \geq 2$, dann ist ebenfalls

$$\mathrm{Aut}\,\Omega(G) = \mathrm{Aut}\,\mathcal{V}(G).$$

Ist $G: F(G) = 4$ und G vom Typ (2,2), dann ist

$$\mathrm{Aut}\,\Omega(G) = S_4 \ , \ \mathrm{Aut}\,\mathcal{V}(G) = S_3.$$

Ist $G: F(G) = 4$ und G vom Typ $(2^n, 2)$; $n \geq 2$, dann ist

$$\mathrm{Aut}\,\Omega(G) = (Z/2Z)^3, \ \mathrm{Aut}\,\mathcal{V}(G) = (Z/2Z)^2.$$

Der Beweis von Satz 2 benutzt wesentlich die Bestimmung des Führers des ganzen Abschlusses von $\Omega(G)$ in seinem totalen Quotientenring.

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J.C. LENNOX: The Fitting-Gaschütz-Hall relation in certain soluble-by-finite groups

Let $\rho_1(H)$, $\rho(H)$, $\psi(H)$ and $\vartheta(H)$ be the Fitting subgroup, Hirsch Plotkin radical, intersection of the centralizers of the chief factors of H, and $\vartheta(H)/\varphi(H) = \rho_1(H/\varphi(H))$ (where $\varphi(H)$ is the Frattini subgroup of H) respectively. For finite groups H $\rho_1(H) = \rho(H) = \psi(H) = \vartheta(H)$ (The Fitting-Gaschütz-Hall relation) and $\rho_1(H)$ is nilpotent. An FGH group is one which satisfies the F-G-H relation and its Fitting subgroup is nilpotent. P.Hall (1961) showed that $\mathfrak{G} \cap (\mathfrak{N}^2\mathfrak{F})$ (f.g. metanilpotent-by-finite) groups are FGH-groups.

Question 1 Are subgroups of $\mathfrak{G} \cap (\mathfrak{N}^2\mathfrak{F})$ groups FGH groups ?

Ex 1 $G \in \mathfrak{G} \cap \mathfrak{N}\mathfrak{A}$ and $H \leq G$ with $\rho_1(H) < \rho(H)$. G does not satisfy Max-n, the maximal condition for normal subgroups and we ask

Question 2 Are subgroups of $\mathfrak{N}^2 \cap$ Max-n groups FGH groups? Not difficult to show that the answer is yes if $H \leq G \in \mathfrak{N}^2 \cap$ Max-n $\Longrightarrow \vartheta(H) \in \mathfrak{N}$.

Theorem 1 $H \leq G \in \mathfrak{G} \cap \mathfrak{A}^2$ (f.g. metaabelian) $\Longrightarrow \vartheta(H) \in \mathfrak{N}$

This result is proved using the classical Nullstellensatz and we finally ask a question which, if it had an affirmative answer, would answer Question 2 for $\mathfrak{G} \cap \mathfrak{A}\mathfrak{N}$ groups

Question 3 If $\Gamma \in \mathfrak{G} \cap \mathfrak{N}$ and A is a Noetherian Γ-module and if $a \in A$, $x \in \Gamma$ are such that for B a Γ-submodule of A of finite index in A we have $a(x-1)^n \in B$ $n = n(B) > 0$ then is there an $N > 0$ with

$$a(x-1)^N = 0 \text{ ?}$$

J. MENNICKE: Lineare Gruppen über Zahlringen

Es wurde berichtet über ältere Arbeiten von H.Bass, J.Milnor, J.P.Serre, M.Newman, T.Kulato und dem Verfasser sowie über neuere noch unveröffentlichte Arbeiten von L.I.Wasserstein und R.Zimmert.

Alle Arbeiten mit Ausnahme der letztgenannten beziehen sich auf das Kongruenzuntergruppenproblem in SL_n über algebraischen Zahlringen. Die Arbeit von Wasserstein bringt eine abschließende quantitative Behandlung der Gruppen SL_2 über Zahlringen mit unendlich vielen Einheiten.

R.Zimmert beweist ein schönes und tiefdringendes Resultat über Bianchi-Gruppen, d.h. Gruppen SL_2 mit ganzen Koeffizienten aus einem imaginär-quadratischen Zahlkörper.

G. MICHLER: Bounds for the central radical of a group algebra

Let G be a finite group and F an arbitrary field of characteristic $p > 0$ dividing $|G|$, and let FG be the group algebra of G over F. Then grouptheoretical upper and lower bounds are given for the F-vector space dimension of the Jacobson radical of the center of a block ideal B of FG involving the structure of the defect group of B and its embedding in G.

S. MORAN: 4-th dimension subgroup

A survey of some results in the theory of Magnus rings of formal power series which lead to the solution of the dimension subgroup problem for $n \leq 4$.

K. NAKAMURA: Quasinormalteiler einiger p-Gruppen

Sei G eine p-Gruppe und N Quasinormalteiler von G, dessen Zentrum Z(N) zyklisch ist. Es wird dann gezeigt, daß ein Element n mit der Ordnung p von Z(N) einen minimalen Quasinormalteiler von G erzeugt.

P.M. NEUMANN: Some terrible non-Hopf groups

This lecture cone in two acts. First it recalled the description of a 1-generator module M over Z_pA, where $A = C_2 \mathrm{wr}\, C$, such that $M \cong V \oplus M$ where V is a simple Z_pA module. The construction gives M as a subcartesian power of V invariant under the endomorphism $\epsilon : V^N \longrightarrow V^N$ where $(v_1,v_2,v_3\ldots)\epsilon = (v_2,v_3,\ldots)$. This part of the lecture was intended to give the basic ideas involved in constructions described by Dr.G.M.Tyrer. The second act was a survey of her results.

She makes (i) A finitely generated group G with $G \cong G \times S$ where S is one of Ruth Camm's simple groups; (ii) A finitely generated group H with $H \neq 1$, $H \cong H \times H$; (iii) Finitely generated Hopf groups A,B such that $A \times B = A \times B \times S$ (S as above).

A.RAE: Fixed point theorems for finite soluble groups and Sylow theory in infinite groups

Theorem Let A be a subgroup of order p^k in the finite p-soluble group G. Then if the p-length of G is greater than $4 \cdot 2^k$. A is contained in two distinct p-subgroups of G.

The proof uses refinements due to Hartley of Dades Fitting chains to construct a fixed point for A or a on a suitable p'-section of G. This result then gives a more direct proof of a theorem of Hartley about Sylow theory for infinite groups.

Let D_π denote the class in which maximal π-subgroups (Sylow π-subgroups) are conjugate. Let $\mathfrak{X}_\pi$ denote the class of locally finite groups G posssessing a Sylow π-subgroup P such that for every local system Σ of finite subgroups of G there is a subsystem Σ_1 of Σ into which P reduces.

Theorem If G is π-serial and locally finite-soluble then $G \in D^s_\pi$ if and only if $G \in \mathfrak{X}^s_\pi$. If G is generated by π-elements then G is the finite extension of a poly-locally nilpotent group.

A.H. RHEMTULLA: Right Ordered Groups

I. A group G is right-ordered (RO) if it can be totally ordered in such a way that for any a,b,c in G, $a < b$ implies $ac < bc$. Any group which has a system with torsion-free abelian factors is in RO. We produce classes of torsion-free groups which are not in RO. For example, $G \notin RO$ if G/G' is finite and G has a nilpotent subgroup of finite index in G. It is known that Z(G) has no (non-trivial) zero divisors if $G \in RO$. Also a locally RO-indicable group is in RO. The problem of finding classes of torsion-free groups, not in RO, but whose integral group rings have no zero-divisors is wide open.

II. Any partial right order can be extended to a right-order in a torsion-free locally nilpotent group.

This answers the question raised by J.C.Ault, 'Extensions of Partial Right Orders on Nilpotent Groups', J.London Math.Soc. (2) (1970), 749-752.

L.RIBES: <u>A Kurosh subgroup theorem for free pro-C-products of pro-C-groups</u>

Let $\underline{C}$ be a class of finite groups closed under the formation of subgroups, homomorphic images and extensions. A pro-$\underline{C}$-group is a projective limit of groups in $\underline{C}$. Let $(X,*)$ be a Hausdorff pointed topological space, and let $\{A_x \mid x \in X\}$ be pro-$\underline{C}$-groups so tha (i) $A_* = 1$, and (ii) the mapping $x \longmapsto A_x$ from $X \setminus \{*\}$ to $\{A_x \mid x \in X\}$ is locally constant. Then we define the concept of a pro-$\underline{C}$-group which is the free pro-$\underline{C}$-product of the groups A_x. This extends the definitions of free pro-$\underline{C}$-group and free product due to Iwasawa, Serre, Neukirch, Gildenhays-Lim, etc. The main result is a structure theorem for open subgroups of this type of free products, along the lines of the Kurosh theorem for discrete groups. As one consequence we obtain that an open subgroup of a free pro-$\underline{C}$-group $F(X,*)$ on a pointed topological space is again a free pro-$\underline{C}$-group on a pointed, compact totally-disconnected space.

J.E.ROSEBLADE: <u>Absolutely Capital Group Rings</u>

A field is absolute if its non zero elements are roots of unity. A capital of a ring is a simple (ring)image. A ring is Jacobson if the Jacobson radical of every image is nilpotent. An absolutely capital Jacobson ring is one J which is commutative, Noetherian, Jacobson and all of whose capitals are absolute.

<u>Theorem.</u> Let H be polycyclic by finite and J an absolutely capital Jacobson ring. Let S = JH.

(i) The capitals of S are all finite dimensional over suitable capitals of J;

(ii) If S* is a (ring) image of S then Im S* is nilpotent.

This result was proved. The method was also used to show

Result If M is a simple JH-module. Then the cantralizer K is an absolute field. If $\dim_K M$ is finite then M is finite dimensional over some capital of J.

R.SANDLING: Integral group rings of finite nilpotent groups

There is a condition on the ideals of the integral group ring of a finite nilpotent group, which would imply both the integral group ring conjecture for finite nilpotent groups and the dimension subgroup conjecture. My previious work on class 2 groups examplifies this condition. A further axample is the case of groups of all upper triangular matrices (with one's on diagonal) over a finite ring; this example provides further bounds on the possible failure of the dimension subgroup conjecture via group representations.

P.E.SCHUPP: Small Cancellation Theory

This talk surveyed some results of "small cancellation" theory. The theory studies quotient groups $G = {}^F/N$ where F has a free product structure and N is the normal closure of a subset $R \subseteq F$ where cancellaeion between elements of R is, in a suitable sense, "small". The theory is a powerful tool for showing decision problems to be soluble and for proving embedding theorems. Recent results illustrating these two sides of the theory are:

Theorem (C.Weinbaum,K.Appel, P.Schupp): The group G of any (tame) alternating knot has soluble conjugacy problem.

Theorem (C.F.Miller III, P.Schupp): Any countable group G can be embedded in a two-generator, complete, hopfian group H_G in a manner preserving being finitely presented. If G has no elements of some finite order $n > 1$, then H_G can also be chosen to be co-hopfian.

D.SEGAL: <u>Automorphism group of Restricted Wreath Product</u>

Let G be the restricted wreath product of a group B by a group X where $|x| > 2$, and let Γ denote the automorphism group of G. If Z is the cdntre of the base group A of G, then

$$N_\Gamma(ZX) = C_\Gamma(A) \rtimes C_\Gamma(X) \rtimes N_\Gamma(X) \cap C_\Gamma(B)$$

and $C_\Gamma(A) = \mathrm{Der}(X,Z)$, $C_\Gamma(X) = \mathrm{Aut}_X(A)$, $N_\Gamma(X) \cap C_\Gamma(B) \cong \mathrm{Aut}(X)$.

Suppose Der $(X, Z_i/Z_{i-1})$ consists of inner derivations for each i, where $Z_i = \zeta_i A$. Then if B is nilpotent,

$$\Gamma = A^* \rtimes C_\Gamma(X) \rtimes N_\Gamma(X) \cap C_\Gamma(B)$$

where $*: G \longrightarrow \Gamma$ is the inner automorphism map.
Now $Z_i/Z_{i-1} = (\zeta_i B/\zeta_{i-1}B) \otimes ZX$; we have the following result:

<u>Lemma:</u> Let X satisfy (i) $\exists\ z \in X$ such that

$o(z) = \infty = |C_X(z) : C_{<zx>}(z)|$.

(ii) X is generated by a subset S such that for each $s \in S$ there are elements $x_1, \ldots, x_n$ of infinite order in X with $1 = [z,x_1] = [x_1,x_2] = \ldots = [x_{n-1},x_n] = [x_n,s]$.

Then for any abelian group D, Der $(X, D \otimes ZX)$ consists of inner derivations.

B.ERDMANN, P.HILTON, U.STAMMBACH: <u>Über die Homologie von zentralen Gruppenextensionen</u>

Es sei $E: N \overset{i}{\rightarrowtail} G \overset{p}{\twoheadrightarrow} Q$ eine zentrale Erweiterung. Die Multiplikationsabbildung $m: G \times N \longrightarrow G$ induziert $m_*: H_*(G \times N) \longrightarrow$ $\longrightarrow H_*G$, wo $H_*- = \{H_n-\}$ den ganzzahligen Homologiefunktor bezeichnet. Da $H_*(G\ N)$ das Tensorprodukt $H_*G\ \ H_*N$ enthält, erhalten wir eine Abbildung $\ : H_*G \otimes H_*N \longrightarrow H_*G$. Für die Erweiterung $\tilde{E}: N \rightarrowtail N \twoheadrightarrow 1$ ergibt dies das Pontryagin-Produkt in H_*N. Im allgemeinen definiert μ eine Modulstruktur in H_*G über dem Ring H_*N . Definiere die <u>Gruppe der unzerlegbaren Elemente</u> $I = \{I_n\}$ durch

$$I_n = \mathrm{coker}\Big(\mu : \bigoplus_{\substack{p+q=n \\ q \geq 1}} H_pG \otimes H_qN \longrightarrow H_nG\Big).$$

<u>Satz.</u> (a) $I_1 = Q_{ab}$, unabhängig von der Extension E.

(b) Beschreibt $\xi \in H^2(Q,N)$ die Extension E und bezeichnet $\Delta : H^2(Q,N) \longrightarrow \mathrm{Hom}(H_2Q,N)$ den Epimorphismus des Theorems der universellen Koeffizienten, dann ist

$$I_2 = \ker(\Delta(\xi)) .$$

Definiere eine Stammerweiterung (P.Hall) als eine zentrale Erweiterung E mit $p_*: G_{ab} \xrightarrow{\sim} Q_{ab}$. Eine Stammerweiterung heißt eine Stammüberlagerung (Schur'sche Darstellungsgruppe), wenn $p_*: H_2G \longrightarrow H_2Q$ die Nullabbildung ist.

Korollar: Sei E eine Stammerweiterung. E ist eine Stammüberlagerung $\Longleftrightarrow I_2 = 0$.

In gewissen Fällen ist auch eine Beschreibung (bis auf Gruppenextensionen) von I_3 möglich. - Die Beweise verwenden die Hochschild-Serre-Spektralreihe für die Erweiterung E, insbesondere die exakte Sequenz

$$G_{ab} \otimes N \xrightarrow{\gamma} H_2G \xrightarrow{p_*} H_2Q \xrightarrow{\Delta(\xi)} G_{ab} \xrightarrow{p_*} Q_{ab} \longrightarrow 0 .$$

S.E.STONEHEWER: Some o-subgroups of $SL(2,Z_{p^n})$

A subgroup H of a group G is permutable (or quasinormal) in G (write $H \mathrel{\text{℘}} G$) if $HK = KH = \langle H,K \rangle$ for all subgroups K of G. Then it is known that if $H \mathrel{\text{℘}} G$ and if H is core-free, then H is a subdirect sum of finite nilpotent groups. Examples of non-normal, permutable subgroups are rare and difficult to construct. Only recently has it been shown that the nilpotency class of a core-free, permutable subgroup of a finite p-group can be arbitrarily large. The present work exhibits this fact within the groups $SL(2,Z_{p^n})$, as n increases, by observation rather than construction. As a Corollary, there is a non-soluble (infinite) group $G = HK$ with $H,K \mathrel{\text{℘}} G$ and H,K metabelian. (By earlier results, a permutable subgroup is always ascendant, even in at most $\omega+1$ steps. Perhaps non-soluble products of two ascendant soluble subgroups are rare.) However, in a more positive direction, a join G of any number of soluble subgroups, each permutable in G, is locally soluble.

H.WIELANDT: Complements of nilpotent Hall factors

Let $H_0 \trianglelefteq H \leq G$ where the indices $|H:H_0|$ and $|G:H|$ are finite and relatively prime. A complement C of H/H_0 in G is a subgroup C of G such that $G = CH$ and $H_0 = C \cap H$. The question whether such a C exists can be reformulated as a question on permuta ion representations of G: A necessary and sufficient condition is that there should be a homogeneous G-space on which H/H_0 acts regularly. In order to construct such a space put, if H/H_0 is abelian,

$$\mathcal{R} := \{R \mid R \leq G\ ;\ \forall_{g\in G}\ |Hg \cap R| = 1\}$$

$$R \sim S \ := \prod_{\substack{r\in R\\ s\in S\\ Hr=Hs}} rs^{-} \in H_0 \qquad (R,S \in \mathcal{R})$$

Then $\Omega = \mathcal{R}/\sim$ is a H-left and G-right space: $\Omega = {}_H\Omega_G$, and H/H_0 acts regularly from the left on Ω. One can

(a) make G act on Ω from the left whenever H and H_0 are normal in G. This leads to the Schur-Zassenhaus theorem for abelian normal Hall subgroups;

(b) make H/H_0 act regularly on Ω from the right whenever $hR \sim Rh$ for all $h \in H$ (and some, hence all, $R \in \mathcal{R}$). This leads to the focal subgroup theorem of D.G.Higman.

Extension to the case of nilpotent Hall factors is easy and leads to the theorem of Frobenius and Grün-Hall. A possibly new application is the Theorem: Let H/H_0 be a nilpotent Hall factor of G. Then the following statements are equivalent:

(a) There is a normal complement for H/H_0 in G;

(b) For every $g \in G$, there is a normal complement of H/H_0 in the group generated by H and g.

Karsten Johnsen, Kiel