MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Distributionen und partielle Differentialgleichungen

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Leitung: Prof. J. Wloka, Kiel   Prof. Z. Zieleźny, New York

Teilnehmer

Bengel, G., Kaiserslautern
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Floret, K., Kiel
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Lysik, B., Wroclaw
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Newberger, E., Buffalo
Rus, I.A., Cluj
Schmets, J., Liège
Slemrod, M., Providence
Trèves, F., New Brunswick
Zaidman, S., Montreal

Vortragsauszüge

Rus, I.A.: Some fixed point theorems for mappings defined on a cartesian product.

Let $(X_i, |\cdot|\cdot_i, i = 1,2)$ be locally convex spaces. Some fixed point theorems are given for a mapping $f : X_1 \times X_2 \longrightarrow X_1 \times X_2$. The theorems are then used in order to discuss solvability of certain initial value problems and boundary value problems.

Fenske, Chr. : Ein Stabilitätsproblem für oberhalbstetige Abbildungen.

Ist $U \subset \mathbb{R}^n$ offen, f eine oberhalbstetige Abbildung von U in die azyklischen Teilmengen von $U, D \subset U$ offen und die Fixpunktmenge F von $f|D$ kompakt, so definieren wir einen lokalen Fixpunktindex $\operatorname{ind}(U,f,D)$ als das Bild von $1 \in H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$

in der folgenden Sequenz

$$H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \xrightarrow[d^*]{} H^n(D \times U, D \times U \setminus \delta) \xrightarrow[i]{} H^n(\gamma f, \gamma f | D \setminus F) \xrightarrow[\pi^{*-1}]{}$$
$$\xrightarrow[\pi^{*-1}]{} H^n(D, D\setminus F) \xrightarrow[\cong]{} H^n(S^n, S^n \setminus F) \longrightarrow H^n S^n,$$ wobei δ die

Diagonale, $d : D \times U \longrightarrow \mathbb{R}^n$, $x,y \longrightarrow x-y$

$\gamma f := \{(x,y) \in D \times U \mid y \in f(x)\}$, $\pi : \gamma f \longrightarrow D$, $(x,y) \longrightarrow x$ und $i: \gamma f \to D \times U\}$ die Inklusion

bezeichnet. Diese Definition läßt sich sodann auf größere Raumklassen erweitern. - Ist f eine oberhalbstetige Abbildung von X in die Potenzmenge von X, so heißt ein Fixpunkt x_o von f abstoßend bezüglich einer Umgebung U von x_o, wenn es zu jeder Umgebung V von x_o ein $n_o \in \mathbb{N}$ gibt, so daß für alle $n \geq n_o$ $f^n(X\setminus V) \subset X\setminus U$. - Wir benutzen die oben skizzierte Definition des Fixpunktindex, um Sätze über die Existenz von nicht-abstoßenden Fixpunkten von oberhalbstetigen Abbildungen mit azyklischen Bildern zu beweisen.

J. Schmets: <u>On properties weaker than barrelledess or evaluability</u>

A locally convex topological vector space E is σ-barrelled (resp. σ-evaluable) if every countable bounded subset of E_s^* (resp. E_b^*) is equicontinuous and E is d-barrelled (resp. d-evaluable) if every countable union of equicontinuous subsets of E^* is equicontinuous whenever it is bounded in E_s^* (resp. E_b^*).

First we show that the properties, we just introduced, are distinct and also distinct from the usual ones of ultrabornological, bornological, barrelled, evaluable and Mackey-spaces.

Then we show that it is possible to put a finer topology Q on E such that (E,Q) still has the same property as E.(Joint work with De Wilde).

Finally if X is a completely regular and Hausdorff space and if $C_P^b(X)$ is the space of all bounded continuous functions on X equipped with a system of semi-norms P, we give characterizations on X and P for $C_P^b(X)$ to have one of the properties listed above, and we give moreover the characterization of the Q-spaces associated to $C_P^b(X)$ for these properties.

Friedman, A.: <u>The Cauchy problem for degenerate parabolic equations</u>

Consider a second order degenerate parabolic operator L. The present paper is concerned with the uniqueness of solutions of the Cauchy problem: $Lu = f$ in a strip $0 < t \leq T$, $u(0,x) = \phi(x)$ for all x in R^n. It is proved that there is at most one solution subject to a growth condition on $|u|$ which depends on the degeneracy of L. In case L is ultra-parabolic, it is sufficient to assume only one-sided growth on u.

Bierstedt, K.-D.: <u>Compactly regular inductive limits and the approximation property.</u>

(The results mentioned in this talk were obtained together with R. Meise, Mainz).

<u>Def.:</u> Let the (Hausdorff) locally convex (l.c.) space E be the l.c. injective inductive limit of the system $\{E_\alpha, i_{\alpha\beta}\}_{\alpha \in A}$ (i.e. the E_α are linear subspaces of E). Then E or $\underset{\vec{\alpha}}{\text{ind}}\, E_\alpha$ is said to be compactly regular iff for each compact $K \subset E$ there is an $\alpha \in A$ such that K is contained and compact in E_α.

Every countable strict or compact inductive limit of complete l.c. spaces is compactly regular. An example of a different type is given by:

Thm. 1: Let X be locally compact and $\mathfrak{V} = \{v_i\}_{i \in \mathbb{N}}$ with continuous functions $v_i > 0$ on X, $v_{i+1} \leq v_i$. Assume that for every i there exists a $j = j(i) > i$ such that v_j/v_i vanishes at infinity. Then the inductive limit $\mathfrak{V}C(X) = \operatorname{ind}_{i \to} Cv_i(X)$ of the Banach spaces $Cv_i(X) = \{f \in C(X)$ (i.e. continuous on X); $\|f\|_i = \sup_{x \in X} v_i(x) |f(x)| < \infty\}$ is compactly regular (and complete).

Thm. 2: Let the quasi-complete l.c. space E be the compactly regular inductive limit of $\{E_\alpha, i_{\alpha\beta}\}_{\alpha \in A}$. If now all the E_α have the approximation property (a.p.), so does E.

Cor.: $\mathfrak{V}C(X)$ as in Thm. 1 has the a.p.

Some generalizations, other examples (mainly of analytic functions) and applications to the ε-tensor product are indicated.

Zaidman, S.: Weak solutions of differential equations in reflexive Banach spaces.

Let X be a reflexive Banach space, X^* its dual. Let $A: D(A) \subset X \to X$ be a linear closed operator with dense domain and A^* its adjoint. For $-\infty \leq a < b \leq +\infty$ define the class $K_{A^*}(a,b)$ of functions $\varphi(t)$ in $C^1(a, b; X^*)$ with compact support in (a,b) such that $\varphi(t) \in D(A^*)$ and $(A^*\varphi): (a,b) \to X$ is continuous. Let $f(t) \in L^p_{loc}(a,b;X)$ be given. The class $W^{(a,b)}_{A,f}$ is composed of functions $u(t) \in L^p_{loc}(a,b;X)$ verifying the relation

$$\int_a^b \langle \varphi'(t) + (A^*\varphi)(t), u(t) \rangle \, dt = -\int_a^b \langle \varphi(t), f(t) \rangle \, dt$$

for any $\varphi(t) \in K_{A^*}(a,b)$. We have a few results, as for example:

Theorem 1 . Let $u(t), f(t) : (a,b) \to X$; be continuous , also $u(t) \in W_{A,f}^{(a,b)}$, $u(t) \in D(A)$ and $(Au) : (a,b) \longrightarrow X$.
Then $u(t) \in C^1(a,b;X)$.

Theorem 2 . Let $f(t) \in C(a,b;X)$, $u(t) \in C^1(a,b;X) \cap W_{A,f}^{(a,b)}$; then $u(t) \in D(A)$ for $a < t < b$.

Grudzinski, O.v.: Über exponentielles Wachstum von Fundamentallösungen bei Convolutions-Gleichungen.

Es werden diejenigen Convolutionsoperatoren $f \in \mathcal{E}'$ charakterisiert, die Fundamentallösungen E $(f \times E = \delta)$ mit exponentiellem Wachstum haben:

Def.: a) $f \in (\mathcal{Y}_\varepsilon) \Leftrightarrow f$ hat Fund. Lösg. $E \in \mathcal{D}'$ mit $E/\cosh(\varepsilon|\cdot|) \in \mathcal{Y}'$

b) $f \in M_{K,\varepsilon} \Leftrightarrow$ es ex. $N, C > 0$: $\sup\limits_{\substack{|\eta| \le \varepsilon \\ \eta \in \mathbb{K}^n}} |\hat{f}(x+\eta)| \ge \frac{C}{(1+|x|)^N}, x \in \mathbb{R}^n$

wobei: $\varepsilon > 0$, $\mathbb{K} = \mathbb{C}, \mathbb{R}$

Satz: Für $f \in \mathcal{E}'$ sind die folgenden sechs Aussagen äquivalent:

(i): $f \in (\mathcal{Y}_\varepsilon)$ für ein $\varepsilon > 0$

(ii): $f \in (\mathcal{Y}_\varepsilon)$ für alle $\varepsilon > 0$

$(iii)_{\mathbb{K}}$: $f \in M_{\mathbb{K},\varepsilon}$ für ein $\varepsilon > 0$; $\mathbb{K} = \mathbb{C}$ oder $\mathbb{R}$

$(iv)_{\mathbb{K}}$: $f \in M_{\mathbb{K},\varepsilon}$ für alle $\varepsilon > 0$; $\mathbb{K} = \mathbb{C}$ oder $\mathbb{R}$

Bem.: Nennt man $\hat{f}$, wenn f $(iii)_K$ oder $(iv)_{\mathbb{K}}$ erfüllt, "extremely slowly decreasing", ergibt sich mit den Ergebnissen von Ehrenpreis (1959) und Hörmander (1962) das folgende Schema:

(1) f hat eine Fund. Lösung $\in \mathcal{D}'$ <=> $\hat{f}$ slowly decreasing.

(2) f hat eine Fund. Lösung endlicher Ordnung <=> $\hat{f}$ very slowly decreasing.

(3) f hat Fund. Lösung von beliebig exponentiellem Wachstum $<=>$ $\hat{f}$ extremely slowly decreasing .
Falls $\hat{f}$ extremely slowly decreasing, werden für f: proper fundamental solutions konstruiert, die die Bed. (ii) des Satzes erfüllen. Für die Differential-Differenzen-Operatoren wird diese Konstruktion verbessert; auch die Bed. $(iv)_{\mathbb{R}}$ wird in diesem Fall in einer verschärften Form bewiesen.

Mitrović, D.: Distributions et problems aux limites des fonctions analytiques.

Problème 1 (Plemelj) Soit T une distribution donnée dans $\mathcal{E}'(\mathbb{R})$. Trouvez une fonction $\hat{T}(z)$ localement holomorphe dans $\mathbb{C}$ coupée le long de Supp T et admettant sur $\mathbb{R}$ la condition au bord
$\hat{T}^{+} - \hat{T}^{-} = T$ où $\hat{T}^{\pm} \equiv \lim_{\varepsilon \to +0} T\,(t \pm i\varepsilon)$ dans D' .
On suppose $\hat{T}(\infty) = 0$
La solution (unique): $\hat{T}(z) = \frac{1}{2\pi i} \langle T_{\tau}, \frac{1}{\tau - z} \rangle$.
Problème 2 (Hilbert). Soit $t \longrightarrow \sigma(t) \nmid 0$ une C^{∞}-function complexe donnée sur $\mathbb{R}$ avec la propriété H au point $t = \infty$.
Soit S une distribution donnée dans $\mathcal{E}'(R)$. Trouvez une fonction $\Phi(z)$ localement holomorphe dans $\mathbb{C}$ coupée le long de Supp S et admettant la condition au bord

$$\Phi^{+} = \sigma(t)\Phi^{-} + S \quad \text{sur } \mathbb{R} .$$

La solution de l'équation de convolution

$$a(t)\, T - \frac{b(t)}{\pi i} (T * vp\, \frac{1}{t}) = S$$

est étroitement liée à la solution du problème de Hilbert.

Trèves, F.: On the existence and regularity of solutions of linear partial differential equations.

In recent years the study of linear partial differential operators $P(x,D_x) = P_m(x,D_x) + P_{m-1}(x,D_x) + \ldots$ has been greatly facilitated by the introduction of and the systematic analysis in the contangent bundle $T^*\Omega$(P is defined in the C^∞ manifold Ω) . A good example of this is the solvability theory for operators of <u>principal type</u> , i. e. , such that $P_m(x,\xi) = 0$ $,\xi \neq 0, \Longrightarrow d_\xi P_m(x,\xi) \neq 0$. The latter property enables us to apply the implicit function theorem in a neighbourhood Γ of a point (x_o,ξ^o) of the characteristic set of P , i.e. such that $P_m(x_o,\xi^o) = 0$: we may write $P_m(x,\xi) = Q(x,\xi)(\xi_N - \lambda(x,\xi'))$ with $\xi' = (\xi_1,\ldots,\xi_{N-1})$ ($N = \dim \Omega$; we have possibly renamed the coordinates). We may then take Γ <u>conic</u>, i.e. stable under ξ-dilations, and Q,λ homogeneous in Γ , of degrees $m-1$ and 1 respectively. By using suitable cut-off functions of the kind $g(x,\xi) \in C^\infty(\Gamma)$ (Γ is open), $g(x,\rho\xi) = g(x,\xi)$, $\forall\rho > 0$, we may reduce the problem of solving $P(x,D)u = f$ near x_o to that of solving a finite number of first-order <u>pseudodifferential</u> equations $D_N v - \lambda(x,D')v = f_1$. The next step is to get rid of $(\mathrm{Re}\lambda)(x,D')$. This is done by applying Egorov's theorem - by performing a canonical tranformation which maps $\tau - \mathrm{Re}\,\lambda(x,\xi')$ into a new coordinate σ (and preserves the fundamental symplectic form $d = \sum_{j=1}^{N} \xi_j \, dx^j$) . This replaces $(\mathrm{Im}\,\lambda)(x,D')$ by a new pseudodifferential operator $B(x,D')$ and we are reduced to the study of the "evolution equation"

$$\partial_N v + B(x,D')v = f_2 , \tag{1}$$

where $B(x,D')$ is essentially self-adjoint. The solvability of (1) follows from and is essentially equivalent to the property that the function $x^N \longmapsto B(x,\xi')$ does not change sign. We may go back to the symbol $P_m(x,\xi)$ of P and restate this property (roughly) by saying that $\mathrm{Im}\,P_m$ does not change sign along the null-bicharacteristic strips of Re P_m . This is the solvability condition (P) .

Newberger, E.: A general Theorem on Hypoellipticity.

Let Ω be a non-empty open subset of R^n and P a polynomial in R^n . We call a distribution $T \in D'(\Omega)$ strongly regular with respect to the differential operator P(D) if to every open set $\Omega' \subset\subset \Omega$ there exists an integer $m \geq 0$ depending on Ω' such that $P^k(D)T$, $k = 0,1,\ldots$, are all of order $\leq m$ in Ω' . We denote by $\varepsilon_P(\Omega)$ the linear space of all distributions in Ω which are strongly regular with respect to P(D) .

Consider now a differential operator W(D) (with constant coefficients) and two spaces $\varepsilon_P(\Omega)$, $\varepsilon_Q(\Omega)$ corresponding to the differential operators P(D) , Q(D) respectively. The operator W(D) is said to be (P,Q)-hypoelliptic if, for any open set $\Omega \subset R^n$, every solution $u \in \varepsilon_P(\Omega)$ of the equation

$$W(D)u = 0 \qquad (1)$$

is in $\varepsilon_Q(\Omega)$,

We prove the following

Theorem. The differential operator W(D) is (P,Q)-hypoelliptic if and only if the polynomials P,Q and W satisfy one of the equivalent conditions:

(I) $Q(\zeta)$ is bounded on every set of $\zeta \in C^n$ where $W(\zeta) = 0$ and both $P(\zeta)$ and $\mathrm{Im}\zeta$ are bounded.

(II) There are constants $\gamma, C > 0$ such that

$$|Q(\zeta)|^\gamma \leq C(1 + |P(\zeta)|)(1 + |\mathrm{Im}\zeta|), \zeta \in \mathbb{C}^n, W(\zeta) = 0 .$$

Slemrod, M.: Asymptotic behavior of nonlinear contraction semigroups

Let us consider as motivation the ordinary differential equation (Cauchy problem)

(1) $\dot{x} + Ax = o$, $x(o) = x_o$

where $x \in R^n$, $x_o \in R^n$, and A is an $n \times n$ matrix. In this case it is well known that the unique solution to (1) is given by $x(t) = \phi(t)x_o$ where $\phi(t)$ is (the principle matrix solution) e^{-At} and $\phi(t)$ satisfies

(i) $\phi(\cdot)\cdot : R \times R^n \to R^n$ continuous

(ii) $\phi(t) \circ \phi(s) = \phi(t+s)$

(iii) $\phi(o) = I$.

This theory has been abstracted to general Banach spaces. In this case B is a Banach space, $A : D(A) \subset B \to B$ is a closed dense defined operator, and the resolvent of $-A$ $(-A + \lambda I)^{-1}$ satisfies a boundedness condition. In this a solution to (1) is provided by the classical Hille-Yosida Phillips Theorem $x(t) = T(t)x_o$ where $T(t)$ satisfies (i),(ii),(iii).

Finally we consider the nonlinear Cauchy problem posed for nonlinear partial differential equations, where A is a nonlinear operator. $D(A) \subset X$ - Hilbertspace. In this case the Cauchy problem $\dot{x} + Ax \ni o$ has a unique solution , $\bar{x}(t) = S(t)x_o$ where $S(t)$ satisfies (i),(ii),(iii) and $\|S(t)x - S(t)y\| \leq \|x - y\|$ if and only if A is a maximal monotone operator in the sense of Browder, Minty, etc..

Kalik, C.: Lösung einiger Randwertaufgaben mit der semidiskreten Methode.

Es werden Ergebnisse gegeben, die sich auf die Näherungslösung einiger Randwertaufgaben mit Hilfe der semidiskreten

Methode beziehen. Die Funktionen, die für die Semidiskretisierung angewendet werden, sind Splinefunktionen. Das führt zu einer ziemlich schnellen Konvergenz derApproximationsfolge Ferner wird die Stabilität der Methode untersucht.

Marcinkowska, H.: <u>Elliptic boundary value problems with distributional data</u>

It is known that the closure of an elliptic boundary value problem posed in a bounded domain Ω of R_n yields a family of homeomorphism between suitably constructed Hilbert spaces, which in the simplest case of an homogeneous boundary value problem are subspaces of a Sobolev space or are adjoined to them. Our purpose is to show that every such homeomorphism defines another boundary value problem. The differential equation is the same as in the basic problem, but the boundary conditions have the form of integro-differential operators and the boundary data are distributions on $\partial\Omega$.

Körner, J.: <u>Darstellung vektorwertiger Distributionen durch Integrale</u>

Für die Grundräume $E := W_{M,a}, E_M, \mathcal{L}, W_{M_i}^{N^i}, E_{M_i}^{N^i}, W_M^N, E_M^N, E^N\ S(m), S(a), S^{(b)}$ (Wloka,Mitjagin) werden Darstellungen $T(\phi) = \sum_{\alpha \in I} \int f_\alpha(x) D^\alpha \phi(x)dx$ $\forall\ \phi \in E$ für die Distributionen $T \in L(E,F)$ angegeben. Beim Beweis werden die entsprechenden Formeln für $F = \mathbb{C}$ und die Nuklearität der Grundräume benutzt. Die Darstellungen liefern Aussagen über das Wachstum der Distributionen und ihre Ordnung.

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Björck, G.: Beurling distributions and partial differential equations

Most of the theory of partial differential equations given in the book by Hörmander is carried through in the more general framework of Beurling distributions. These are defined in the same way as Schwartz distributions but using a more restricted class of test functions f, with the extra condition given on the Fourier transform side: $\int_{R^n} |\hat{f}(\xi)| \exp(\lambda\ \omega(\xi))\ d\xi < \infty$ for every $\lambda > 0$. Here ω is a given subadditive function, which in the Schwartz case could be taken to be $\log(1+|\xi|)$. It is also proved that, the famous "equation without solution" found by Hans Lewy could be given a right hand member f which is as regular as a restricted testfunction and still the equation does not have a solution of the "wild" generalized distribution kind. See ARK. MAT. 6 (1966), 351 - 407.

Bojarski, B.: Asymptotic solutions of PDE

The investigation of the asymptotic behaviour of integrals $\int_{\Omega} a(x) \exp(i\lambda g(x))\ dx \quad \Omega \subset R^n$ known under the name: method of stationary phase, is an important tool used in the study of asymptotic behaviour of solutions of partial differential equations depending on a parameter λ. Quasi - classical asymptotic of the solution of the Cauchy problem for the equations of quantum mechanics, asymptotic behaviour of the spectral function of differential equations are the simplest examples.

The fundamental relevance of this method for the study of PDE was fully recognized in the work of Maslov on perturbation theory. He was also the first to give a discussion of the asymptotic of solutions of the Schrödinger equation for large time intervals. Besides other things that discussion involved a new notion of topological character. (Maslov index - or the corresponding cohomology class). Maslov also outlined the applications of this theory to a vast class of problems of mathematical physics and to the theory of propagation of singularities of solutions of PDE. The method of stationary phase is also a method unifying the theory of pseudodiff.-operators, Hörmanders theory of Fourier integral operators and their numerous applications. The independent and important work of Hörmander should help to elucidate some basic problems of Maslov theory and its further developments. These theories involve the use of important notions of classical mechanics - canonical transformations, Lagrangian manifolds etc. They show deep relations to some notions of algebraic topology.

All these theories seem to bring fresh and fruitful ideas into the theory of PDE and they show deep and far reaching perspectives for new areas of research, which will bring light to many problems of the general theory of PDE.

After Maslov and Hörmander important work has been done by Baslaev and Leray. Following their ideas some contributions to the subject were also obtained recently by T. Bałaban, J. Kisyński and the speaker: these are: the notion of the generalized solution of the Cauchy problem and its systematic use, introduction of Lagrangian bundles, detailed elaboration of some connections with representation theory of symplectic groups .

Zieleźny, Z.: On spaces of solutions of partial diff.equations

Let $P(D)$ be a differential operator in R^n with constant coefficients and consider the space $E = \{u \varepsilon C(R^n): P(D)\, u = o\}$ with the topology induced in E by $C(R^n)$.

Komūra and Mitjagin proved that the functional dimension df E of E has the following properties:

(a) For any differential operator $P(D)$, df $E \geq n$

(b) $P(D)$ is hypoelliptic if and only if df $E < \infty$

(c) If every solution $u \varepsilon E$ is in the Gevrey class Γ^{ρ}, $\rho = (\rho_1, \ldots, \rho_n)$, then df $E \leq |\rho| = \rho_1 + \ldots + \rho_n$.

In particular, if $P(D)$ is elliptic then df $E = n$.

Suppose now that $P(\zeta)$ is of degree m and

$$P(\zeta) = \zeta_n^m + Q_1(\zeta_1, \ldots, \zeta_{n-1})\zeta_n^{m-1} + \ldots + Q_m(\zeta_1, \ldots, \zeta_{n-1})$$

For fixed $\xi_1, \ldots, \xi_{n-1} \varepsilon R$ let $\zeta_n = \xi_n + i\, \eta_n$ be a solution of $P(\xi_1, \ldots, \xi_{n-1}, \zeta_n) = 0$.

Further, let $K(r)$ be the number of all (n-1)-tuples of integers ≥ 0 such that

$$|\, P(\xi_1, \ldots, \xi_{n-1}, \xi_n) \,| \leq r^m$$

If for large r , $K(r) \geq r^{\lambda}$

then df $E \geq \lambda$

In particular we obtain the following corollaries:

(1) If $n = 2$ then $P(D)$ is elliptic if and only if df $E = n$

(2) If $m = 2$ then $P(D)$ is elliptic if and only if df $E = n$.

J . Körner (Kiel)