MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

T a g u n g s b e r i c h t 22|1973

Finite Geometries

27. 5. bis 2. 6. 1973

Die diesjährige Tagung "Finite Geometries" stand wieder unter der Leitung von Prof. D.R. Hughes, (London) und Prof. H. Lüneburg (Kaiserslautern). Neben neuen Ergebnissen über projektive und affine Ebenen kamen auch wichtige Resultate über Blockplane zur Sprache. Ferner befaßte sich ein Teil der Vorträge mit Codierungs- und Graphentheorie. Die große Zahl der Teilnehmer wirkte anregend auf den Austausch von Meinungen und Informationen, ohne die persönliche Atmosphäre zu beeinträchtigen.

Teilnehmer

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Th. Beth, Göttingen
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S. Suter, Bonn
H.C.A. van Tilborg, Eindhoven
M. Walker, London
R.W. Williams, London

Vortragsauszüge

J. André: Über endliche Ebenen mit nichtkommutativen Verbindungslinien.

Eine Struktur $(V,L,\sqcup,\parallel)$ mit $V \neq \emptyset$ (Punkte), $L \subseteq \mathfrak{P}V$ (Linien), $\sqcup : (V\times V)\setminus\Delta_V \twoheadrightarrow L$, $(x,y)\mapsto x\sqcup y$ (Verbindungsoperation, die nicht notwendig kommutativ ist) und $\parallel \subseteq L\times L$ (Parallelität) heißt fastaffine Ebene, wenn für sie folgende Eigenschaften gelten: (L1) $x,y\in x\sqcup y$, (L2) $x\sqcup y = x\sqcup z \Leftrightarrow z\in (x\sqcup y)\setminus\{x\}$, (L3) $x\sqcup y = y\sqcup x = x\sqcup z \Rightarrow x\sqcup z = z\sqcup x$ (solche Linien heißen Geraden); (P0) $\parallel$ ist Äquivalenzrelation, (P1) zu $l\in L$ und $x\in V$ gibt es genau eine Linie $l' = x\sqcup y \parallel L$. (P2) $x\sqcup y \parallel y\sqcup x$, (P3) g Gerade, $g\parallel l \Rightarrow l$ Gerade; (G1) es gibt mindestens zwei nichtparallele Geraden, (G2) ist g Gerade und $g \nparallel l$, so gilt $|g\cap l| = 1$. Es werden Beispiele echter fastaffiner Ebenen angegeben, die in Zusammenhang mit Fastkörpern und cartesischen Gruppen stehen, und Eigenschaften insbesondere endlicher fastaffiner Ebenen gebracht.

A. Beutelspacher: On parallelisms in finite projective spaces.

Es wird folgender Satz bewiesen:

(Theorem 2): Sei $d = 2^{i+1} - 1$ $(i=1,2,\ldots)$. Dann besitzt $\Sigma = PG(d,q)$ einen Geradenparallelismus.

Zur Vorbereitung benötigt man den folgenden Spezialfall:

(Theorem 1): Jeder 3-dimensionale proj. Raum $PG(3,q)$ der endlichen Ordnung q besitzt einen Parallelismus.

T. Beth: Algebraic Resolution Algorithms for some infinite Families of 3-Designs.

$$\text{Let } F_q = \begin{cases} GF(q)\cup\{\infty_1\} & \text{if } q \equiv -1(3) \\ GF(q)\cup\{\infty_1,\infty_2\} & \text{if } q \equiv 1(3) \\ (GF(3))^n & \text{if } q = 3^n \end{cases}$$

for any primepower q. Let G_q be the group of all affine transformations of

$\overline{F}_q \setminus \{\infty_1, \infty_2\}$, which can be considered as a permutation group on $\overline{F}_q$.

$$\text{Let } m_q = \begin{cases} 1 & \text{if } q \equiv -1(3) \\ 2 & \text{if } q \equiv 1(3) \\ 25 & \text{if } q \equiv 0(3). \end{cases}$$

Thm: Let n be a positive integer, such there exists a primepower $3n-2 \leq q \leq 3n$. Then there are m_q parallel classes $\mathfrak{A}_1, \ldots, \mathfrak{A}_{m_q}$ in $(\overline{F}_q, P_3(\overline{F}_q))$ and subsets $H_1, \ldots, H_{n_q}$ of G_q with the property, that

$$\bigcup_{i=1}^{r_q} H_i(\mathfrak{A}_i) \text{ is a resolution of } (\overline{F}_q, P_3(\overline{F}_q)).$$

F. Buekenhout: The geometry of lines on a quadric.

The following result was obtained together with E. Shult.
Let S be a finite set of points together with distinguished subsets of cardinality ≥ 3 called lines, such that: for each line L and each point p not on L, p is adjacent to either one or all points of L, two district points being adjacent if there is some line containing both of them. Moreover we assume that there is no point of S adjacent to all other points and that there exists some line in S. Then S belongs to one of the following types:

(1) the set of absolute points and totally isotopic lines of a polarity in some PG (q)

(2) the set of singular points and singular lines of a quadratic form in some PG (q)

(3) the set of points and lines of a generalized 4-gon. The finiteness assumption may be replaced by a "finite rank" condition which leads to a similar classification (with additional types).

P.J. Cameron: Locally Symmetric Designs

In a 2-design D, the line trough two points is the intersection of all blocks trough the two points. D is locally symmetric if, for any point p, the configuration $D^*(p)$ of lines and blocks through p is a symmetric-design (equivalently, if the number of blocks through three non-collinear points and the number of points in two non-disjoint blocks are both nonzero constants.)

Theorem: Let D be a locally symmetric design with parameters v,k,λ, in which a line has s+1 points and three non-collinear points lie in t blocks. Then exactly one of the following is true:

(i) D is the design of points and hyperplanes in a projective Geometry of dimension at least 3 over GF(s);

(ii) D is the design of points and hyperplanes in an affine geometry of dimension at least 3 over GF(s+1), $s > 1$;

(iii) D is a hadamard 3-Design ($s=1$, $v=4(t+1)$, $k=2(t+1)$, $\lambda = 2t+1$);

(iv) $v = (1+st)(2s^2+2s+1+(3s+2)s^2t+s^4t^2)$, $k=(1+st)(1+s+s^2t)$, $= 1+(2s+1)t+s^2t^2$, and if $s > 1$ and $t > 1$ then $t > s+2$ and s+1 divides $t(t-1)$;

(v) $t=1$, $v=(s+1)^4(s^3+2s^2+3s+1)$, $k=(s+1)^2(s^2+s+1)$, $\lambda = s^3+3s^2+4s+3$;

(vi) $s=1$, $t=3$, $v=4qb$, $k=40$, $\lambda=39$.

P. Delsarte: Inner and outer distribution of t-designs

Let V be a finite set of "points", $v = |V| \geq 2$. For an integer $k, 1 \leq k \leq [v/2]$, let D be any nonempty set whose elements (blocks) are k-subsets of V. The inner distribution of D is the (k+1)-tuple $\underline{a} = (a_o,\ldots,a_k)$ where a_j denotes the average number of blocks meeting a fixed block in j points. Then D forms a t-design $S_\lambda(t,k,v)$, for a given $t, 1 \leq t \leq k$, iff $\underline{a}$ satisfies some well-defined linear equations.

Let $x_o,\ldots,x_{m-1}$ be the k-subsets of V, with $m = \binom{v}{k}$. The outer distribution of D is the m (k+1) matrix A where $A_{i,j}$ denotes the number of blocks meeting x_i in j points. Then A^TA is congruent over $\mathbb{Q}$ to the matrix $\mathrm{diag}(b_o,\ldots,b_k)$, the b_j being well-defined linear functions of $\underline{a}$. Hence $b_j \geq 0$, $\forall_j$, which yields, in particular, a lower bound to λ for $S_\lambda(t,k,v)$ with given (t,k,v); for eample, this shows the nonexistence of $S_5(4,8,17)$. Furthermore, one obtains a method to compute A, which is illustrated for $S_1(4,8,24)$.

Michael Doob: Some graph theoretic questions with geometric aspects.

Many of the most useful examples of highly symmetric graphs have arisen from constructions that are based on the various structures of finite geometries. The concept of a strongly regular graph had its genesis in R.C. Bose's study of (r,k,t) geometries. While there has been some effort to generalize the concept of a strongly regular graph to graphs with greater diameter, there have, as yet, been relatively few constructions. Some families of non isomorphic graphs with the same parameters and large diameter will be presented and some open questions that might be of interest to geometers will be discussed.

B. Fischer: D-subgroups of ${}^2E_6(2)$.

Theorem (A. Evers)

Let D be the set of central involutions in $G = {}^2E_6(2)$; let U be a maximal subgroup of G generated by elements of D which is not a local subgroup. Then $U \cong M(22)$, $F_4(2)$, $Sp(8.2)$.

B. Ganter: Equational classes of Steiner systems.

An equational class (that is, a variety) $\mathfrak{A}$ of universal algebras is said to have property (k,m), if any k-generated algebra in $\mathfrak{A}$ has exactly m elements. If $\mathfrak{A}$ is a variety with property (k,m) for $k,m \in \mathbb{N}$, and if $A \in \mathfrak{A}$, then the k-generated subalgebras of A form a Steiner System of type (k,m).

Theorem: An equational class with property (k,m) exists for

(i) k=m

(ii) k=0, m=1;

(iii) k=2, m a primepower;

(iv) k=3, m=4;

and for no other pair (k,m) of natural numbers.

Let q be a primepower, and let A be a primitve element of GF(q). Define

$$[x,y]^i := (\ldots(x\circ \underbrace{y)\circ y)\circ \ldots)\circ y}_{i\text{-times}} .$$

The variety $\mathfrak{R}^q_2$ of quasigroups defined by the equations

$$x \circ x = x$$

$$[x,y]^{q-1} = x$$

$$[x,y]^i = [y,x]^j \quad \text{whenever } a^i + a^j = 1 \text{ in } GF(q)$$

has property (2,q) and many other interesting algebraic properties. Each Steiner System of type (2,q) can be "coordinatized" by a quasigroup in $\mathfrak{R}^q_2$. The additional equations $(a\circ b)\circ(c\circ d) = (a\circ c)\circ(b\circ d)$ or, for $q > 3$, $a\circ(b\circ c) = (a\circ b)\circ(a\circ c)$ and $(a\circ b)\circ c = (a\circ c)\circ(b\circ c)$ lead to a subclass of $\mathfrak{R}^q_2$ coordinatizing all affine spaces of order q, which are desarguesian.

Two applications:

a) The solution to problem 1) in Vortragsbuch 21, p. 181 is yes. The same result holds for Steiner Systems of type (3,4).

b) Let A,B be nonisomorphic Steiner Systems of type (2,q), both of prime cardinality. Then

$$\mathrm{Aut}(A\times B) \cong \mathrm{Aut}(A)\times \mathrm{Aut}(B).$$

This all is joint work with Heinrich Werner.

J.M. Goethals: <u>Binary codes defined by quadratic forms over GF(2).</u>

The set Q of quadratic forms on V(m,2) has an isomorphic representation as an $\binom{m+1}{2}$-dimensional subspace of $V(2^m-1,2)$, known in coding theory as the second order Reed-Muller code RM(m,2). In this representation, the image of the set L of linear forms is the first order Reed-Muller code RM(m,1). Furthermore, Q/L is isomorphic to the set B of bilinear alternating forms on V(m,2). To each form b in B, there corresponds in the above representation a coset of RM(m,1), properly contained in RM(m,2), whose (Hamming) weight distribution is uniquely determined by the rank of b. Then it follows that the distribution of distances in any code

consisting of a union of cosets of RM(m,1), properly contained in RM(m,2), is uniquely determined by the distribution of ranks in the corresponding subset of B.
We shall consider the problem of constructing the best codes in this family.

Armin Herzer: Dualitäten mit zwei Geraden aus absoluten Punkten in projektiven Ebenen.

Auszug aus der gleichnamigen Arbeit, veröffentlicht in MZ 129, 235-257 (1972).

J. Joussen: On the projectivity group in André planes of degree 2.

This is a report on a result of Mr. A. Longwitz (Dortmund).
Theorem A (Longwitz): Let Π be an André plane of degree 2, $\Pi = \Pi(A_q, \omega, 2, \lambda)$ satisfying $n_\lambda > 2e_\lambda$, where $n_\lambda := |\{i \in I_{q-1} \mid (i) = 0\}|$, $e_\lambda := |\{i \in I_{q-1} \mid \lambda(i) = 1\}|$ $(I_{q-1} := \{0, 1, \ldots, q-2\})$. Let X be a line of Π and $\mathcal{P}_x$ the group of all projectivities of X onto itself. Then $\mathcal{P}_x \supset \mathfrak{A}_x$ (the alternating group on X). If q is odd, then $\mathcal{P}_x = \gamma_x$.
Corollary: Let $\Pi = \Pi(H_q)$ be the Hall plane of order q^2, and let $\mathcal{P}_x$ as before. Then $\mathcal{P}_x \supset \mathfrak{A}_x$. If q is odd, then $\mathcal{P}_x = \gamma_x$.
Theorem B (Longwitz): Let Π be an André plane of degree 2, and $\mathcal{P}_x$ as before. Suppose that $\mathcal{P}_x$ is quadruply transitive. Then $\mathcal{P}_x \supset \mathfrak{A}_x$. If the order of Π is odd, then $\mathcal{P}_x = \gamma_x$.
Corollary: Let $\Pi = \Pi(N_{q,2})$ be the regular nearfield plane of degree 2 and order q^2, and $\mathcal{P}_x$ as before. Then $\mathcal{P}_x = \gamma_x$.

Michael J. Kallaher: Rank (k,1) planes.

Finite affine planes with collineation groups transitive on the affine lines are known to be translation planes (Wagner). However this is not true if the collineation group is merely transitive on the points. We consider the following situation: Let Π be a finite affine plane with a collineation group G transitive on the points of Π. Let σ be an affine point of π . A block orbit of G_σ is an orbit Γ of G_σ such that $\Gamma \cup \{\sigma\}$ is the union of lines trough σ . If π is non-square order and G possesses a block orbit of degree $\gamma \geq 2$, then π is a translation plane.

We can associate with G two integers k and l defined as follows: k is the rank of G as a permutation group on the points and l is the number of orbits of G on l_∞, the line at infinity of π. If $l > \frac{k}{2}$, and π has non-square order, then π is a translation plane.

Christiane Lefevre: Generalized quadrangles in projective spaces.

Let P be any finite dimensional projective space. A generalized quadrangle Q in P is a pair (G,L) where G is a set of points of P and L a set of lines of P such that:

(i) every point on a line of L belongs to G

(ii) given any line D of L and any point p of G not belonging to D, there is exactly one line trough p intersecting D.

Q is a thick quadrangle if every line of Q possesses the same number $K \geq 3$ of points and if every point of Q is on the same number $R \geq 3$ of lines.

Non thick quadrangles are easy to describe. Our purpose is to classify the thick quadrangles of a projective space P. We can only reach this goal in finite projective spaces, but most of our arguments are valid also in infinite spaces and it looks reasonable to hope a solution of the problem without the finiteness assumption.

The result, obtained together with Buekenhout, is:

If Q is a generalized quadrangle in a finite projective space P, then Q is one of the following:

(a) a quadric in some $P_4(q)$

(b) an elliptic quadric in some $P_5(q)$

(c) a symplectic quadric in some $P_3(q)$

(d) a hermitian quadric in some $P_3(q^2)$

(e) a hermitian quadric in some $P_5(q^2)$.

J.H. van Lint: A theorem on equidistant codes.

A set C of m binary words of length n at mutual distances 2k is called an equidistant (m,2k,n)-code. If $\hat{C}$ is the matrix which has as its rows all the code words of C then C is called trivial if every column of $\hat{C}$ has m-1 or m equal entries.

Theorem: For $k > 1$ a nontrivial $(k^2+k+2,2k,n)$-code exists (for sufficiently large n) if and only if a PG(2,k) exists. This extends a result of M. Deza who proved that if $m > k^2+k+2$ then the code must be trivial.

Heinz Lüneburg: Rang-3-Ebenen.

Eine affine Ebene heißt Rang-3-Ebene, falls sie eine auf den Punkten transitive Kollineationsgruppe besitzt, so daß der Stabilisator eines Punktes genau drei Bahnen hat. Nach Kallaher und Liebler sind alle solche Ebenen Translationsebenen, falls sie nur endlich sind.

Satz 1. Ist p eine Primzahl mit $p+1 = 2^r$, ist A eine affine Ebene der Ordnung p^2 und besitzt A eine Kollineationsgruppe vom Rang 3, die auf g_∞ eine Bahn der Länge 2 hat, so ist A eine verallgemeinerte Andréebene, es sei denn, es ist $p = 7$.

Zusammen mit Ergebnissen von Kallaher und Ostrom erhält man damit den

Satz 2. Ist A eine endliche Ebene der Ordnung q und besitzt A eine Rang-3-Kollineationsgruppe, die auf g_∞ eine Bahn der Länge 2 hat, so ist A eine verallgemeinerte Andréebene, es sei denn es ist $q = 5^2, 7^2, 11^2, 23^2, 29^2, 59^2, 2^6$.

In all diesen Fällen außer im Falle q = 64 sind Ausnahmen bekannt.

F.J. Mac Williams: Coding Theory and Combinatorial Designs.

It is shown by very elementary methods that the codewords of weight 8 in the extended Golay code form a 5-(24, 8, 1) design. This is a special case of a general theorem: Let C be a code in F^n, and d' the minimal distance of the dual code C . Let $0 < \tau_1 < \dots < \tau_s < n$ be the weights of codewords of C. If $\bar{s} < d'$, the codewords of each weight form a r-$(n, \tau_i, \lambda_i^{(r)})$ design for $1 \le r \le d'-s$, and $\lambda_i^{(r)} \cdot \prod_{j \neq i} (\tau_j - \tau_i) = -\frac{s(n)}{n-{}_i} + \frac{|C|}{2^n} \sum_{t=r}^{n} \binom{n-r}{t-r} \frac{s(t)}{{}_i - t}$,

where $s(t) = \prod_{j=1}^{\bar{s}} ({}_i - t)$.

Rudolf Metz: <u>On finite Wille geometries of grade n.</u>

Wille geometries of grade n are some geometries, such that the derivation through n points is a projective geometry. For example, a Möbius geometry is a Wille geometry of grade 2. Now always assume $n \geq 2$ and dimension $d \geq 3$!! Results of Assmus, Ganter, Wille show that any finite Wille geometry of grade n has a well defined order, which is the order of all irreducible projective derivations. The following theorems hold:

1. (Wille) Any Wille geometry of grade n and dimension $d \geq 4$ can be represented by a point set of a projective geometry.
2. A Wille geometry of grade n and order q is representable in a projective geometry of order q, if it is representable.
3. For a Wille geometry of grade n and order 2, n=2 holds. For any dimension there are exactly three types, constructed from the affine spaces over GF(2).
4. There exists no Wille geometry of grade n and order 3.

Julia M. Nowlin Brown: <u>Homologies Generated by Elations.</u>

<u>Theorem:</u> Let G be a collineation group of a finite projective plane π of odd order n. Let G contain nonidentity elations α, β with their respective centers A,B and axes a,b in general position (i.e. $A \neq B$, $a \neq b$, $A \not I b$, $B \not I a$, AIa, and BIb). Assume that

1) G is linear on AB

or 2) n   3 (mod 4), π has no even order subplane, and if γ and δ are two nonidentity elations in $G_{AB,ab}$ with distinct centers and if $\langle\gamma\rangle P = \langle\delta\rangle P$ then P = ab.

Then G contains an order two homology center ab and axis AB.

C. Norman: <u>'Hadamard Designs'.</u>

Using a construction due to Todd, a large number of non-isomorphic Hadamard designs are constructed, the isomorphism classes being in 1-1 correspondence with a double coset decomposition of a symmetric group. The connection with equivalence classes of Hadamard matrices is also discussed.

N.J. Patterson: On Kerdock codes.

A Kerdock system is a set $\mathcal{f}$ of symplectic forms on a vector space of dimension 2n over GF(2) sucht that

(i) $0 \in \mathcal{f}$

(ii) $S_1, S_2 \in \mathcal{f}$ and $S_1 \neq S_2 \Rightarrow S_1 + S_2 \in \mathcal{f}$

(iii) $|\mathcal{f}| = 2^{2n-1}$.

A construction is given for such a system.

- J.J. Seidel: Quadratic forms over GF(2).

The quadratic forms of V(2m,2) are interpreted as vectors of a real space of dimension 2^{2m}. The inner product of two forms is the Arf invariant of their sum. The Gramian matrix of the quadratic forms (which is blocked according to the alternating bilinear forms) provides a setting for certain combinatorial objects, such as Hadamard matrices, linked symmetric designs, coherent configurations, and binary codes.
(Joint work with P.J. Cameron, to be published in Proc. Ned. Akad. Wetensch.)

M. Walker: On the characterisation of some translation planes.

A proof of the following theorem was obtained:
Theorem: Let A be a translation plane of odd order q^2. Assume that
a) the Kern of A contains a subfield isomorphic to F_q and
b) A admits a collineation group isomorphic to $SL_2(q)$ in its translation complement.

Then: (1) A is desarguesian
or (2) A is a Hallplane
or (3) A is a Hering plane
or (4) A is one of two exceptional planes of order 25.

Planes in (1) and (2) exist for all q and those in (3) only when $q \equiv -1 (\mathrm{mod}\ 6)$.
The four classes are mutually disjoint.

M. Dugas (Kaiserslautern)

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