MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

T a g u n g s b e r i c h t 23/1974

Differentialgeometrie im Großen

2. bis 8.6.1974

Die Tagung stand unter der Leitung von M.Barner und W.Klingenberg. Mit dieser Tagung wurde das neue Vortragsgebäude eröffnet. In einer kurzen Eröffnungsansprache erinnerte Herr Barner daran, daß vor knapp zwei Jahren mit einer gleichnamigen Tagung der alte Lorenzenhof zum letzten Mal genutzt wurde.

Teilnehmer

S.Alexander, Urbana
M.Barner, Freiburg
M.Berger, Paris
J.P.Bourguignon, Paris
K.Chen, Urbana
P.Dombrowski, Köln
H.Duistermaat, Nijmwegen
P.Ehrlich, Stony Brook
H.Eliasson, Reykjavik
J.Eschenburg, Bonn
D.Ferus, Münster
E.Heintze, Bonn
H.C.Im Hof, Bonn
H.Karcher, Bonn
W.Klingenberg, Bonn
S.Kobayashi, Berkeley
J.Lafontaine, Paris
P.Marry, Paris
Meyer, Paris
H.F.Münzner, Bremen
K.Nomizu, Bonn
M.Obata, Bonn
H.Osborn, Urbana
A.Poor, Stony Brook
E.Ruh, Bonn
P.Sentenace, Paris
U.Simon, Berlin
G.Stanilov, Sofia
T.Takahashi, Bonn
R.Walter, Dortmund
F.W.Warner, Philadelphia
T.J.Willmore, Durham
W.Ziller, Bonn

Vertragsauszüge

S.Alexander: Two results in differential geometry of immersions.

The problem is considered under which conditions an isometric immersion of $M_1 \times \ldots \times M_k$ into E^{n+k} of codimension k is a product of hypersurface immersions. This is known to be true if each M_i is compact by a theorem of J.D.Moore. It is wrong in general but shown to still hold if no M_i contains a "minimizing Euclidean strip" isometric to $E^{n_\alpha-1} \times (-\varepsilon,\varepsilon)$. It is a corollary that if each M_i has nonnegative sectional curvature and some point where the conullity of curvature is at least three, then M is rigid in E^{n+k}.

Secondly C^1 immersions of compact connected orientable manifolds M into E^{n+1} are considered with the property that the tangent planes at saddle points, where no neighbourhood of the point lies on one side of the tangent plane, do not exhaust E^{n+1}. It is shown that such an M is homeomorphic to S^n and it is the union of two n-discs with common boundary, each of which is imbedded into E^{n+1}.

J.P.Bourguignon: Poisson formula for riemannian manifolds.

We gave generalizations for a compact C^∞ manifold of the classical Poisson formula for the circle relating the sum of the values of a function at integer points and the same sum for its Fourier transform up to a factor 2π. These results are due to Colin de Verdiere, Chazarain, Duistermaat and Guillemin and relate the spectrum of the laplacian on functions with length, index, and nullity of the closed geodesics by means of an asymptotic expansion in case the closed geodesics form nondegenerate critical submanifolds with respect to the energy function on the free loop space.

K.Chen: Bar construction on de Rham complexes.

Let M be a manifold with a base point, and let A be a differential graded subalgebra of the de Rham complex $\Lambda^{\bullet}(M)$ such the inclusion induces cohomology isomorphismus. For simplicity assume $A^0 = k = \mathbb{R},\mathbb{C}$. Let $B(A)$ be the reduced bar contruction on A. If M is simply connected, then

$$H^{*}(B(A)) \approx H^{\bullet}(\Omega(M),k)$$

where $\Omega(M)$ is the loop space. In some cases $H^{*}(\Omega(M))$ can be computed.

P.Dombrowski: Frenet theory for differentiable maps into riemannian manifolds.

To generalize the well known Frenet theory for curves the following notions are introduced:
The span of the image of the first k derivatives of a map $f : M \to N$ (at each point) is called the k^{th} osculating space.
f is called a Frenet (k,s) map if the k^{th} osculating spaces form an s-dimensional vectorbundle along f and is called a Frenet (∞,s) map if the union of all osculating spaces form such a vector bundle. In the first situation a higher order bilinear fundamental form is defined with values in the normal bundle of the k^{th} osculating space in

the same way as the usual second fundamental form. Typical results are:

If f is Frenet (∞, s) then the image of f lies in a s-dimensional totally geodesic submanifold of N and the same conclusion holds if f is a Frenet (k,s) map with the property that the above bilinear form vanishes.

The machinery of Gauss, Codazzi and Weingarten equations are also developed.

H. Duistermaat. The Morse index in variational calculus.

In the Poisson formula for positive elliptic pseudo-differential operators appears an integer as a power of i which in a special case is identified as the Morse index of a variational problem related to the symbol of the differential operator. This integer was given as the intersection number of a curve of Lagrange spaces with a fixed Lagrange space and by homotopying the curve and changing the fixed space appropriately one shows that this integer is equal to the Morse index, in the cases where this Morse index is finite.

P. Ehrlich: Critical metrics, line fields, and curvature.

Berger observed that for a product metric g_o on $S^2 \times S^2$ any deformation of g_o non-negative at first order for the sectional curvature vanishes identically at first order. We ask whether this behaviour characterizes product metrics calling such metrics (with $Ric \geq 0$ or $K \geq 0$) strongly critical at first order. The method of local convex deformations implies that there exists a direction with 0 Ricci curvature at every point. For a 3-dimensional manifold it follows that for each point p either all Ricci curvatures at p are 0 or there exists a local unit parallel vector field near p. Under certain conditions this implies that locally the manifold splits isometrically into products.

H.Eliasson: Variations of mappings

Let M and M' be compact and connected C^∞ manifolds. Given a C^∞ map $f : M \longrightarrow M'$ does there exist a harmonic map in the homotopy class of f? Eells and Sampson were able to answer this question in the affirmative if the sectional curvature of M' is nonpositive by means of solving the heat equation $\frac{\partial u}{\partial t} = \Delta u$ with $u/_{t=0}=f$ for $0 \leq t < \infty$ and proving uniform convergence to a harmonic map as $t \rightarrow \infty$. We have established a differential inequality for the solution which imply the result of Eells and Sampson and moreover obtain convergence to a harmonic map even if M' has sectional curvature positive and the initial map f satisfies certain energy conditions. An inequality is given under which u converges to a point map.

D.Ferus: Immersions with parallel second fundamental form

Consider an isometric immersion $f : M \longrightarrow E^{n+p}$ with parallel (i.e. covariantly constant) second fundamental form. For $p=1, M=S^k \times E^e$. For higher codimension we classify all such immersions. They come from standard isometric immersions of symetric R-spaces into euclidean space. Similar things can be said if the ambient space has constant sectional curvature.

E.Heintze: Homogeneous manifolds of strictly negative curvature

Let M be a homogeneous manifold of negative curvature. Then M admits a simply transitive solvable group of isometrics, so that M may be represented as a solvable Lie group with left invariant metric.

It is discussed which Lie groups occur and shown that there exist many homogeneous non-symetric manifolds of negative curvature. It is also shown that the non-symetric ones do not admit compact quotients.

H.Hernandez: <u>On compact manifolds with positive Ricci curvature.</u>
Consider the modified Brieskorn manifolds $V'(a_1...a_n) = f^{-1}(0) \cap g^{-1}(0)$ with $f(z_1..z_n) = z_1^{a_1}/a_1(a_1-1) + .. + z_n^{a_n}/a_n(a_n-1)$, $g(z_1..z_n) = z_1 \cdot \bar{z}_1/a_1 + ... + z_n \cdot \bar{z}_n/a_n - 1$

Then we can show that for any $a_1 \geq .. \geq a_n \geq 2$ there exists an $M(a_1..a_n)$ such that for $m \geq M$ $V'(a_1..a_n, a_{n+1}, ..a_{n+m}), a_{n+1} = ..=a_{n+m}=2$ has strictly positive Ricci curvature. It follows especially that there exist infinaltely many exotic spheres which admit a metric of positive Ricce curvature. These examples contain all Kervaire spheres in dim 4k+1.

H.C.Im Hof: <u>An equivariant pinching theorem</u>

Let M be a compact simply connected manifold with positively pinched sectional curvature $0 < \delta \leq k \leq 1$ and G a compact Lie group acting by isometries on M. We prove that there exists a δ_n depending on the dimension of M with $\delta_n \to 0{,}7$ as $n \to \infty$ so that for a δ_n pinched manifold there exists a diffeomorphism $g : M \to S^n$ and a homomorphism $\varphi : G \longrightarrow O(n+1)$ such that g is φ-equivariant. It follows that a non-simply connected manifold with δ_n pinched sectional curvature is diffeomorphic to some space of constant sectional curvature. This work was done together with E.Ruh.

H.Karcher: Jacobi field methods and Lie groups

Writing a Jacobi field J along a geodesic c with left translations $J(s) = d(L_{c(s)})A(s)$ one can give an explicit formula for $A(s)$ and apply this to solve $\exp A \cdot \exp tB = \exp H(A,tB)$ giving a geometric prove of the Campbell-Hausdorff formula. From this follows: Define a Finsler metric on the group by left translating an ad G-invariant norm, then the left invariant geodesics also minimize Finsler distances and balls of radius $< \frac{\pi}{2}$ are convex. The center of mass depends on an estimate for $\| J(s) - s J'(s) \|$ and a description of the construction by Jacobifields. Applications are given: Conjugation of subgroups and improvement of almost homomorphism to homomorphism.

W.Klingenberg: Generically there are infinitely many closed geodesics

Let M be a compact simply connected manifold and $G(M)$ the space of Riemannian metrics of class C^k, $k \geq 5$, with the C^k topology. Then there exists a residual subset of $G(M)$ such that all metrics in this subset have infinitely many prime closed geodesics. This set is defined as follows: The Poincaré map P associated to the closed geodesic c shall have no eigenvalue $= 1$ and the exponents $\frac{\alpha}{2\pi}$ of the eigenvalues $e^{i\alpha}$ of P shall be independent over the rationals, and this condition has to be satisfied for all closed geodesics of the metric. A theorem of Klingenberg and Takens implies that this set is residual. The proof that these metrics have infinitely many closed geodesics uses the sequence of non-trivial rational homotopy classes of the space $\Lambda(M)$ of closed curves on M constructed by P.Sullivan. It also uses the equivariant Morse theory on $\Lambda(M)$ and a formula for the index of an iterated closed geodesic due to Bott.

S.Kobayashi: <u>On complex Finsler Geometry</u>

The purpose of the talk was to show that it is interesting and important to consider complex Finsler metrics. There are some natural examples of Finsler metrics which arise in other areas of mathematics. For example, if E is a holomorphic vector bundle of rank r over M and L is the so called tautological line bundle over

$P(E) = E - \{\text{zero section}\}/C^*$, then $E - \{\text{zero section}\} = L - \{\text{zero secti[illegible]}\}$

(More precisely, L is obtained from E by blowing up the zero section, thus $L = (E - \{\text{zero section}\} \cup P(E))$. Hence there is a natural one to one correspondence between the hermitian fibre metrics in L and the Finsler fibre metrics in E. Furthermore the dual of E is ample if E admits a Finsler fibre metric of negative "curvature". The (infinitesimal) Caratheodory metric of a complex manifold M and the infinitesimal version of the intrinsic distance are also complex fibre metric.

J.Lafontaine: <u>Some properties of the Jacobi variety of a riemannian manifold</u>

The "carcan" of a compact orientable Riemannian manifold (X,g) is the lowest bound of the volumes of those submanifolds of X of cod[illegible] which are not homologous to zero. The question is, if the function

$g \mapsto \dfrac{vol(X,g)^{n-1}}{carcan(X,g)^n}$ $(n = \dim X)$ on the space of Riemannian metrics on X has a positive lower bound. For $n=2$ the answer is yes (Loewner and Blatter). The Jacobi variety $J(X,g)$ of (X,g) is the flat Riemannian torus of dimension $b = \dim H^1(X, R)$ equal to the harmonic 1-forms divided by the harmonic 1-forms with integer periods. Then

1) There exists $u(b) > 0$ such that

$$\frac{\mathrm{vol}(X,g)^{n-1}}{\mathrm{carcan}(X,g)^n} \geq u(b)\cdot \varphi(g),\ \varphi(g) = \frac{\mathrm{vol}(X,g)^{\frac{n-2}{2}}}{\mathrm{vol}(J(X,g))^{n/b}}$$ for every g (M.Berger)

2) If $n \geq 3$ and $0 < b \leq n$, the function $g \mapsto \varphi(g)$ has no critical points unless $X = T^n$. In this case the critical points are all the flat metrics but the Hessian of φ at these points is never positive semi-definit.

P.Marry: <u>The problem of intersection in Riemannian geometry</u>

Given an r-plane bundle $E \xrightarrow{\pi} X$ over a n-dimensional manifold X equiped with a fiber metric and a connection ∇ compatible with this metric, one can construct over the unit sphere bundle $S \xrightarrow{\pi_S} X$ of E a so-called relative area element which is a (n-1) form $\mathcal{b}$ over S such that $\int_{\pi_S} \mathcal{b} = 1$ and $d\mathcal{b} = -\pi_S^*(x)$, x being the Euler form of E with respect to ∇. This form $\mathcal{b}$ can be used to compute some kind of residue calculus around intersections of submanifolds of a given manifold.

K.Nomizu: <u>Geodesics on submanifolds</u>

1) Let M^n be a submanifold of E^m. Assume each geodesic on M^n is a circle in E^m. If the codimension is 1 or 2 then M^n is a euclidean n-sphere. For higher codimension the (real) Veronese variety $S^n \longrightarrow E^m$ $m = {}^{n(n+3)}/2$, and the Marnnoury embedding $P^n(\mathbb{C}) \longrightarrow E^m$ have the same geometric property without being umbilical. our geometric condition implies that the second fundamental form is parallel, so that a recent result of D.Ferus can be used.

2) A curve x_s in M^n,parametrized by are length, is called a circle if there exists a field of unit vectors Y_s and a constant

$k > 0$ such that

$$\nabla_s \vec{x}_s = kY_s \quad , \quad \nabla_s Y_s = -k\vec{x}_s$$

For an isometric immersion $f : M^n \longrightarrow \tilde{M}^m$ each circle in M^n is mapped onto a circle in M^m if and only if f is umbilical and has parallel mean curvature vector.

3) Let M^n be a Kähler submanifold of the complex projective space $P^m(\mathbb{C})$. If each geodesic on M^n is contained in some projective line $P^1(\mathbb{C})$ then M^n is totally geodesic. If each geodesic in M^n is a circle in $P^m(\mathbb{C})$ then either M^n is totally geodesic or M^n is the Veronese variety.

M.Obata: <u>On a certain system of differential equations on a riemannian manifold.</u>

Sometimes a riemannian manifold is completely determinded by the existence of a solution for a system of differential equations.

<u>Theorem:</u> Let M be complete connected riemannian n-manifold $(n > 1)$ Then M admits a nontrivial solution of the equation

$$\nabla_j \nabla_i f + c^2 f g_{ji} = 0 \ ; \ c > 0$$

or of the equation

$$\nabla_k \nabla_j f_i + c^2(2f_k g_{ji} + f_j g_{ik} + f_i g_{kj}) = 0, \ f_i = \nabla_i f, \quad c > 0$$

if and only if M is isometric to a euclidean sphere of radius $1/c$. Similar theorems are proved for the complex and quaternion projective space.

H.Osborn: <u>Constructions of primary classes</u>

Let $\mathfrak{a}$ be the de Rham algebra of a manifold M, and for any complex n-plane bundle ξ over M let F be the $\mathfrak{A}$-module induced by the smooth sections of ξ. Then there is a product in $E = \oplus \mathrm{End} \Lambda^r F$ for which the trace is an algebra homomorphism onto $\mathfrak{A}$. It follows that if $Z(\xi)$ (resp. $B(\xi)$) consist of those elements of E whose trace is closed (resp. exact) then $Z(\xi)/B(\xi)$ is a non-trivial algebra $A(\xi)$ over H (M, R) with a trace induced algebra projection $A(\xi) \longrightarrow H^*(M, R)$. The curvature $K \in \mathrm{End}\, F$ of any connection $D : F \longrightarrow F$ in ξ yields elements of $A(\xi)$ independently of the choice of D, which project to the usual Chern classes of ξ. There is an analogous factorization of the Grothendieck construction of Chern classes.

A.Poor: Convex sets in Riemannian geometry

The geometry of a Riemannian manifold is related to the properties of its convex sets and convex functions. As an example, the proof of the Soul Theorem uses the existence of a naturally exhaustive family of totally convex sets in a complete nonnegatively curved manifold. To make the constructions more natural it would be desirable to have some sort of differentiable approxiamation lemmas for convex sets and convex functions.

U.Simon: A further method in global differential geometry

Let M be a connected oriented Riemannian manifold and A a Codazzi-tensor on M (i.e. symetric (1,1) tensor satisfying the Codazzi equations). We prove a Simons type equation for such tensors:

$$\frac{1}{2}\left(\|A\|^2\right) = \|\nabla A\|^2 + \langle \nabla(\text{grad trace } A), A\rangle + \sum_{i<j} K_{ij}(\lambda_i - \lambda_j)^2$$

where λ_i are the eigenvalues of A with eigendirections X_i and

K_{ij} are the sectional curvatures with respect to the X_i, X_j planes. Applications of this formula always imply $\nabla A = 0$ under certain conditions. Applications are given to the theory of submanifolds in euclidean space, infinitesimal bendings, conformally flat and conformally symetric manifolds, and to homogeneous manifolds.

T.Takahashi: On the transnormality of homogeneous hypersurfaces in a sphere

Consider the linear isotropy representation of a symetric Riemannian manifold of the compact type $I_o(M)/_K$ Thus K is a subgroup of $O(n+1)$ and the orbit $K(A)$, $A \in R^{n+1}$ is a homogeneous submanifold of $S^n \subset R^{n+1}$. $K(A)$ has codimension 1 in S^n if the rank of the space is 2 and according to a theorem of Hsiang and Lawson these are all homogeneous hypersurfaces of S^n. We study such submanifolds of S^n (where the rank is arbitrary now) and express the principal curvatures and their multiplicity to the root system of the associated orthogonal symetric Lie algebra. We also prove that every such $K(A)$ is transnormal.

R.Walter: Gauss-Bonnet for geodesically convex sets

Let C be a compact locally convex subset of the Riemannian manifold M. The following Gauss-Bonnet formula holds:

$$\int_C \gamma = a_k \cdot \int_{\partial C^r} \left[f^* R^k \wedge F \wedge (DF)^{m-2k-1} \right] + \chi(c), \ r > 0 \text{ small}$$

Here γ = Gauss-Bonnet-Chern integrand, $a_k > 0$ the usual universal numbers, C^r = outer parallel set of C in distance r, R = curvature operator, $f : \partial C^r \to \partial C$ metric projection, $F : \partial C^r \to T_1 M/_{\partial C}$ with

F(q) = unit tangent vector of the unique minimal geodesic from f(q) to q. There is an application of this formula stating if $K \geq 0$ along ∂c for $m \leq 6$ or curvature operator ≥ 0 along ∂c for $m \geq 2$ then a Cohn-Vossen type inequality holds:

$$0 \leq \int_c \gamma \leq \chi(c)$$

F.W.Warner: Prescribing curvature

This is joint work with Jerry Kazdan. It is proved that the obvious Gauss-Bonnet sign condition is sufficient for a smooth function on a given compact 2 manifold to be the Gaussian curvature of some metric and that any function negative somewhere on a compact manifold M of dimension 3 is the scalar curvature of some metric and if M admits constant positive scalar curvature then every smooth function on M is the scalar curvature of some metric. The older method of proving some of these results is by conformal deformations. More recently we have applied the method of Fischer and Marsden directly analyzing the map $F(g) = K$ from metrics to Gaussian or scalar curvature. This method is simpler but does not imply as much information as conformal deformations. We have also applied this method with partial results to Euler forms or integrands.

T.J.Willmore: k-harmonic manifolds

Let M be an analytic pseudo-Riemannian manifold. In normal coordinates at p_o there is a (1,1) tensor given for every p in this neighbourhood by $t^i_j = g^{ik}_p \, (g_{kj})_o$. Let Z_k be the elementary symetric polynomial of degree k of the eigenvalues of $t^i j$. Then we define M as k-harmonic at p_o if Z_k involves p_o and p only in terms of $\Omega = \frac{1}{2}er^2$, where r is the geodesic distance from p_o to p and e is the signature of the metric. We say M is k-harmonic if it is k-harmonic at all $p \in M$. It has been shown that n-harmonic implies 1-harmonic and some results have been given that suggest that the converse is also true. This may help to settle the conjecture that a n-harmonic manifold with positive definte metric is locally symetric.

W.Ziller (Bonn)