Mathematisches Forschungsinstitut Oberwolfach

Tagungsbericht 25/1976

Algebraische Gruppen

13.6. bis 19. 6. 1976

Unter der Leitung der Professoren T. A. Springer und J. Tits fand in der Woche vom 13. 6. bis zum 19. 6. 1976 eine Tagung über "Algebraische Gruppen" statt. Der vorliegende Tagungsbericht enthält eine Zusammenstellung von Vortragsauszügen. Dabei erschien es aus inhaltlichen Gründen sinnvoll, die für die Tagung gewählte Vortragsreihenfolge in diesem Bericht beizubehalten.

Teilnehmer

Abe, E., Tokyo
Bartels, H.-J., Göttingen
Behr, H., Frankfurt
Borel, A., Princeton
de Concini, C., Rom
Haboush, W. J., Princeton
Harder, G., Bonn
Helverson-Pasotto, A., Paris
Hesselink, W. H., Groningen
Humphreys, J. E., Amherst
Hurrelbrink, J., Bielefeld
Jantzen, J. C., Bonn
Kneser, M., Göttingen
Kraft, H., Bonn
Lehrer, G., Sydney
Luna, D., Grenoble
Lusztig, G., Coventry
Mostow, G. D., New Haven
Procesi, C., Rom
Rehmann, U., Bielefeld

Rentschler, R., Paris

Richardson, R. W., Durham

Rousseau, G., Orsay

Soulé, C., Paris

Spaltenstein, N., Coventry

Springer, T. A., Utrecht

Stuhler, U., Göttingen

Tits, J., Paris

Veldkamp, F. D., Utrecht

Vortragsauszüge

T. A. Springer: Representations of Weyl groups and nilpotent elements

Let G be a complex connected semi-simple Lie group, with Lie algebra $\mathfrak{g}$. Fix a nilpotent element $A \in \mathfrak{g}$, and denote by Z(A) its centraliser in G, by C(A) the quotient of Z(A) by its identity component and by B_A the variety of Borel subgroups whose Lie algebra contains A. One knows that

$\dim B_A = \frac{1}{2}(\dim Z(A) - \operatorname{rank}(G))$.

One can construct representations of the Weyl group W of G (relative to some maximal Forms) in the graded cohomology of B_A with constant coefficients say in an algebraically closed field E of characteristic 0. Actually one even has a representation of $C(A) \times W$ in $H^*(B_A, E)$. For each irreducible representation φ of C(A), denote by $H^{2\dim B_A}(B_A, E)_\varphi$ the corresponding isotypic subspace.

Theorem: If $H^{2\dim B_A}(B_A, E)_\varphi \neq 0$, the representation of $C(A) \times W$ in this space is of the form $\varphi \otimes \chi_{A,\varphi}$, where $\chi_{A,\varphi}$ is an irreducible representation of W. Each irreducible representation of W arises in this manner, for some pair (A, φ) which is unique up to conjugacy.

Corollary: Suppose that the following condition is satisfied: for any nilpotent $A \in \mathfrak{g}$, the irreducible representations of $C(A)$ are defined over $\mathbb{Q}$. Then all irreducible representations of W are defined over $\mathbb{Q}$.

The construction of Weyl group representations also works over an algebraically closed field of not too small characteristic $p > 0$, working in l-adic cohomology with $l \neq p$. They can be used to give a character formula on unipotent elements for the representations of finite Chevalley groups of Deligne - Lusztig.

G. Lusztig: Characters of principal series of finite Chevalley groups.

Let G be a split semisimple group over F_q. An irreducible complex representation R of G is in the principal series if it occurs in the representation induced by the unit representation of a Borel subgroup. It is known that there exists a class function on the Weyl group $f : W \longrightarrow \mathbb{Z}$ such that the value of R at any regular semisimple element of type w in $G(F_q)$ equals f(w). The main result of this talk was that f is independent of q, in an obvious sense. Moreover, f can be defined directly in terms of the complex Lie group. The proof uses results of Deligne concerning complete reducibility of monodromy over an algebraic space of parameters.

G. I. Lehrer: Regular Unipotent Elements and Gaussian Sums.

Let G be a connected reductive group over k, the algebraic closure of F and suppose G defined over F_q, and has a Frobenius endomorphism F. We are concerned with the following problem: given a regular unipotent element x of G^F and an irreducible complex cuspidal character ρ of G^F what is the value $\rho(x)$? If G has connected centre, then $\rho(x) = (-1)^s (\rho, \Gamma)$ (=0 or $(-1)^s$, s = F_q-rank) where Γ is the Gelfand-Graev character of G^F. The proof of this depends on the fact that if B = TU is an F-stable decomposition of an F-stable Borel subgroup B, then T^F acts transitively on the regular linear characters of U^F. When the centre of G is not connected, this is not true and the orbits of regular linear characters are parametrized by $H^1(F,Z)$ the first Galois cohomology group (G semi-simple). By embedding G in $\bar{G} = G \times \bar{Z} / \{(\zeta, \zeta^{-1}) \mid \zeta \in Z\}$ where $\bar{Z}$ is a torus containing Z, one sees that the T^F-orbits of linear characters in $\Gamma/_{G^F}$ form a coset of a certain well-defined subgroup $Y(\rho)$ of $H^1(F,Z)$. Moreover for G simply connected Z cyclic and split one can show that the orbit sums σ_ζ ($\zeta \in H^1(F,Z)$) are given by the formula $\sigma_\zeta = \sum_{\psi \in H^1(F,Z)^\wedge} \overline{\psi(\zeta)}\, g_{\psi^{n_1}} g_{\psi^{n_2}} \cdots g_{\psi^{n_\ell}}$ where the n_i are integers depending only on the root system of G and for $\phi \in H^1(F,Z)^\wedge$, g_ϕ is the Gaussian sum $\sum_{s \in F_q} \phi(s)\chi(s)$, χ a fixed character of F_q^+. The remaining case of Spin(4n) can be dealt with similarly.

J. E. Humphreys: On the hyperalgebra of a semisimple algebraic group.

Let G be a simply connected semisimple algebraic group over an algebraically closed field K of characteristic $p > 0$. Following a suggestion of D.-N. Verma, we study the structure of the "hyperalgebra" $U_K = U_{\mathbb{Z}} \otimes K$, where $U_{\mathbb{Z}}$ is the Kostant $\mathbb{Z}$-form of the corresponding complex enveloping algebra. For each power $q = p^r$ there is a canonical subalgebra $\underline{\underline{u}}_r$ of U_K having dimension $q^{\dim G}$, $\underline{\underline{u}}_1$ being the restricted universal enveloping algebra of Lie(G). The irreducible $\underline{\underline{u}}_r$-modules are derived from the irreducible G-modules of highest weight $\lambda = \sum c_i \lambda_i$ $(0 \leq c_i < q)$, and the largest of these (the Steinberg module St_r) is also a projective module. Thanks to the Hopf algebra structure of U_K, it is possible to form tensor products of $\underline{\underline{u}}_r$-modules; in particular, tensoring with St_r yields a projective $\underline{\underline{u}}_r$-module. In this framework we obtain a quick representation-theoretic proof of the main step in W. J. Haboush's proof of the Mumford conjecture: reductive groups are geometrically reductive.

William J. Haboush: Report on Vanishing Theorems

The talk was a summary and description of recent results on the vanishing of higher cohomology of line bundles on generalized flag varieties and on generalized Schubert cells. Results of

Kempf, L. Bai, Musilli and Seshadri, Demazure, and H. Anderson were described, and methods were reported upon. Special attention was given to Anderson's result which is the most general. It states that the higher cohomology of dominant line bundles restricted to generalized Schubert cells vanishes and that the restriction map on sections is surjective.

Claudio Procesi: Invariant Theory

We prove the first fundamental theorem for invariants under SL(n,K), K an infinite field or the integers, of m-vectors $x_1,\ldots,x_m$ and m-covectors $\varphi_1,\ldots,\varphi_m$ i. e. every invariant is a polynomial in $\langle\varphi_i,x_j\rangle$, $[x_{i_1},\ldots,x_{i_n}]$, $[\varphi_{i_1},\ldots,\varphi_{i_n}]$.
Corollary: The centralizer of GL(V) acting on $V^{\otimes m}$ is spanned by the symmetric group S_m, V a finite dim. vector space over a field K with at least m elements, or a finite free Z-module.

R. W. Richardson: The conjugating representation of a semi-simple group.

Let G be a simply connected semisimple algebraic group over an algebraically closed field k of char(o) and A(G) the algebra of regular functions on G. Let C(G) be the algebra of regular class functions on G. Theorem: There exists a G-stable vector subspace H of A(G) such that the product map $C(G) \otimes_k H \to A(G)$ is a linear isomorphism. In particular A(G) is a free C(G)-module.

D. Luna: Groupes réductifs de transformations

On exposera quelques faits généraux sur les groupes réductifs de transformations: soit G un groupe algébrique réductif opérant morphiquement dans une variété algébrique affine lisse X, le corps de base étant algebriquement clos et de caractéristique nulle; l'opération de G dans X, au voisinage de toute orbite fermée T, ressemble alors beaucoup à l'opération de G dans le fibré normal à T, au voisinage de la section nulle.

A. Borel: Stable cohomology of $SL_n(Z)$ and regulators

Let $\underline{G}$ be a connected semi-simple group defined over $\underline{\underline{R}}$, $G=\underline{G}(\underline{\underline{R}})$, K a maximal compact subgroup of G, G_u a maximal subgroup of $\underline{G}(\underline{\underline{C}})$ containing K, $X=G/K$ and $X_u = G_u/K$, and Γ a discrete subgroup of G. Using G-invariant differential forms on X, one defines a natural homomorphism $j_\Gamma : H^*(X_u; \underline{\underline{R}}) \longrightarrow H^*(\Gamma ; \underline{\underline{R}})$. The problem discussed in this talk was the effect of j_Γ on rational cohomology, mainly in the case where $\underline{G} = R_{k/\underline{\underline{Q}}} SL_n$, where k is a number field, and $\Gamma = SL_n\underline{\mathcal{O}}$, where $\underline{\mathcal{O}}$ is the ring of integers of k, in the stable range where j_Γ is an isomorphism. This leads to the regulators R_m appearing in Lichtenbaum's conjectures in algebraic K-theory. In particular R_m is equal, up to a rational functor, to $\lim_{s \to -m} \frac{\zeta_k(s)}{(\pi(s+m))^{d_m}}$, where ζ_k is the zeta-function of k and d_m the order of the zero of ζ_k at $-m$ $(m \in \underline{\underline{N}}, m \neq 0)$.

G. Harder: Cohomology of $\Gamma \subset SL_3(\mathbb{Z})$

Die Kohomologie von einer Kongruenzuntergruppe $\Gamma \subset SL_3(\mathbb{Z})$ ist gleich der Kohomologie des Quotienten X/Γ wobei $X = SO(3) \backslash SL_3(\mathbb{R})$. Dieser Quotient kann nach Borel-Serre kompaktifiziert werden - $X/\Gamma \hookrightarrow \overline{X}/\Gamma$ -, so daß $\overline{X}/\Gamma$ eine Mannigfaltigkeit mit Ecken ist. In diesem Fall tauchen Randmannigfaltigkeiten der Dimension 4 - diese sind offen - und Randkomponenten N_1 der Dimension 3 auf. Es wurde über ein Theorem von J. Schwermer berichtet, das eine Beschreibung des Bildes

$$H^{\bullet}(X/\Gamma, \mathbb{C}) = H^{\bullet}(\overline{X}/\Gamma, \mathbb{C}) \longrightarrow \bigoplus_{1} H^{\bullet}(N_1, \mathbb{C})$$

liefert.

J. Hurrelbrink: Endlich präsentierte arithmetische Gruppen über Laurent-Polynomringen

Sei k ein endlicher Körper, $k[t,t^{-1}]$ der Laurent-Polynomring in einer Variablen über k.
Während die Gruppe $SL_2(k[t,t^{-1}])$ ein Beispiel einer endlich erzeugten, nicht endlich präsentierten arithmetischen Gruppe liefert, soll etwa mittels Behandlung der Gruppen SL_n für $n \geq 3$ gezeigt werden, daß für halbeinfache Chevalley-Gruppen G ohne einen Faktor vom Typ A_1 (oder G_2) die arithmetischen Gruppen $G(k[t,t^{-1}])$ endlich präsentiert sind.
Entsprechendes gilt für die Gruppen SL_n bei hinreichend großem n über Laurent-Polynomringen in mehreren Variablen.

U. Rehmann: Endlich präsentierte Matrizengruppen

Satz 1: Sei A endlich erzeugte kommutative Z-Algebra mit 1, mit Krull-Dimension d und $n \geq d+3$. Sind dann $K_1(A), K_2(A)$ endlich erzeugt, so ist $GL_n(A)$ endlich präsentiert.

Beweis mittels des folgenden Resultats von Van der Kallen:

$n \geq d+3 \Rightarrow K_2(n,A) \simeq K_2(A)$ und

Satz 2: A wie in Satz 1, Φ reduziertes irreduzibles Wurzelsystem, $St(\Phi,A)$ die zugehörige Steinberggruppe mit Koeffizienten in A. Ist dann $rk\ \Phi \geq 3$, so ist $St(\Phi,A)$ endlich präsentiert.

U. Stuhler: Über die Frage der endlichen Präsentierbarkeit der SL(2) im Funktionenkörperfall

Sei K ein algebraischer Funktionenkörper vom Transzendenzgrad 1 über dem endlichen Körper k, S eine endliche Menge von Punkten des zu K gehörigen glatten vollständigen Modells Y,

$O_S := \bigcap_{\mathfrak{y} \notin S} O_{Y,\mathfrak{y}}$ ein Ring ganzer Größen.

Für die Gruppe $\Gamma := SL(2,O_S)$ gilt dann

1.) Γ nicht endlich erzeugt $\Longleftrightarrow |S| = 1$

2.) Γ nicht endlich präsentierbar $\Longleftrightarrow |S| \leq 2$

Speziell ist Γ endlich präsentierbar, falls $|S| \geq 3$ ist.

C. Soulé: Reduction theory on affine buildings

Let $\underline{G}$ be a Chevalley group over $\mathbb{Z}$, K the field $\mathbb{F}_q(t)$, Γ the group $\underline{G}(\mathbb{F}_q[t])$, I the affine Bruhat-Tits building associated to the valuation $\omega(\frac{a}{b})$ = degb - dega on K. We show that Γ operates on I with the standard "quartier" as a fundamental domain. Furthermore we exhibit a "good" subspace for the action of Γ on X. A few consequences are derived.

E. Abe: Coverings of twisted Chevalley groups over commutative rings

Let A be a commutative ring with 1 and with an involutive automorphism, Φ be a root system of type $A_n (n \geq 2)$, $D_n (n \geq 4)$ or E_6. Then the group $G(\Phi,A)$ of A-valued points of a simply connected Chevalley-Demazure group scheme G of type Φ has naturally an involutive automorphism σ and let $G_\sigma(\Phi,A)$ be the subgroup of $G(\Phi,A)$ consisting of the elements fixed by σ which we call a twisted Chevalley group over A. We shall define elementary subgroup $E(\Phi_\sigma,A)$, of $G_\sigma(\Phi,A)$ which is generated by some unipotent elements of $G_\sigma(\Phi,A)$, give the relations between these elements, define Steinberg group $St(\Phi_\sigma,A)$ of $E(\Phi_\sigma,A)$ and discuss (homological) connectedness and simply connectedness of the groups.

G. D. Mostow: <u>Non-arithmetic lattices</u>

It is known that lattices in semi-simple groups of $\mathbb{R}$-rank greater than 1 are arithmetic. Examples of non-arithmetic lattices in SO(n,1) are known, but so far no examples are known of non-arithmetic lattices in the other $\mathbb{R}$-rank 1 groups: SU(n,1), Sp(n,1), F_4. A construction was described for defining groups in SU(n,1) which leads to a class of groups that are sometimes discrete. Some of these turn out to be arithmetic lattices. One of them is possibly an non-arithmetic lattice; the resolution of the question depends on knowing whether the group is discrete.

Guy Rousseau: <u>Construction of buildings</u>

The purpose of this lecture was to construct the affine building of a semi-simple (connected) algebraic group $\mathfrak{G}$ over a field complete for a real not trivial valuation. We use a theorem of Galois descent proved by Bruhat and Tits in (Pub. IHES # 41 §9) and the existence of such building in case $\mathfrak{G}$ is split and we show the theorem: "if $\mathfrak{G}$ has a building over a tamely ramified Galois extension of K, then $\mathfrak{G}$ has a building over K". Whence $\mathfrak{G}$ has a building when $\mathfrak{G}$ quasi-splits over a tamely ramified Galois extension, including the case when the valuation is discrete and the residuefield perfect (Bruhat-Tits).

Moreover Tits has proved the existence of the building for all semi-simple groups except some forms of the exceptional groups, and we can increase the relative rank of these forms with extensions of degree 2, so it follows that every semi-simple group has a building if the residual characteristic is not 2. By the same methods we prove that if $\mathfrak{g}$ is K - anisotopic, then $\mathfrak{g}$ (K) is bounded.

N. Spaltenstein: All components of B_u have the same dimension

Let B be a Borel subgroup of a reductive algebraic group G. Let C be the class of a unipotent element $u \in B$. Then there exists a natural relation between the components of B_u and the components of $C \cap B$. Using this relation I prove that all components of B_u have the same dimension and that all components of $C \cap B$ have the same dimension.

H. Kraft: Conjugacy classes in sl_n

Let g be a semisimple Lie algebra /$\mathbb{C}$ and let $\mathfrak{p} \subset g$ be a parabolic subalgebra. Then the sheet $\mathcal{J}_{\mathfrak{p}}$ is defined to be the union of conjugacy classes Gx (G adjoint group, $x \in \mathfrak{r}_{\mathfrak{p}}$ = solvable radical of $\mathfrak{p}$) of maximal dimension (= $2 \cdot \dim \mathfrak{n}_{\mathfrak{p}}$, $\mathfrak{n}_{\mathfrak{p}}$ = nilpotent radical of $\mathfrak{p}$). The sheets are in one - to - one correspondence to the subsets of a basis B of the rootsystem R of $\mathfrak{g}$ up to conjugation under the Weylgroup W of R. Thm. sl_n is the disjoint union of sheets (Ozeki - Wakimoto - Dixmier). Hence the space sl_n/PSl_n of orbits in sl_n has a stratification given by the subsets $\mathcal{S}_\varphi/PSl_n$ of orbits in the sheet $\mathcal{S}_\varphi$.

If $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan - subalgebra of $\mathfrak{g}$, $\mathfrak{h}_\varphi = \mathfrak{h} \cap \mathfrak{r}_\varphi$, we can associate to any element $h \in \mathfrak{h}_\varphi$ an orbit $\phi(h)$ in $\mathcal{S}_\varphi$: $\phi(h) = G(h + \mathfrak{n}_\varphi^h) \cap \mathcal{S}_\varphi$. Let W be theWeylgroup of $\mathfrak{g}$ and $W_\varphi = \mathrm{Norm}_W(\mathfrak{h}_\varphi)$. <u>Thm.</u>: ($g = sl_n$) : ϕ induces a homeomorphic map $\bar{\phi}$: $\mathfrak{h}_\varphi/W_\varphi \xrightarrow{\sim} \mathcal{S}_\varphi/PSl_n$ and $\mathfrak{h}_\varphi/W_\varphi \xrightarrow{\sim} A^{d_\varphi}$, $d_\varphi = \dim \mathfrak{h}_\varphi$

F. D. Veldkamp: <u>Regular elements and characters in finite Chevalley groups</u>

Let G be an almost simple linear algebraic group, σ an endomorphism of G such that G_σ, the group of fixed points of σ, is finite, and T a σ-invariant maximal torus of G. An element $t \in T$ is <u>regular</u> if $\alpha(t) \neq 1$ for all roots α, and <u>in general position</u> if $w(t) \neq t$ for all $w \neq 1$ in the Weyl group of T. Similar definitions for characters of T. We discuss the existence of elements in general position and of regular elements in T_σ, and similarly for characters.

A. Borel: <u>Conjugacy classes in real semi-simple groups</u>

Let G be a connected semi-simple real linear group, K a maximal compact subgroup of G. Let $C_K = K/_{\mathrm{Ad}\ K}$ be the space of conjugacy classes of K and $\sigma_K : K \longrightarrow C_K$ the canonical projection. An extension

$\mathfrak{b} : G \longrightarrow C_K$ of $\mathfrak{b}_K$ which is a class function was defined, and the main part of the talk was devoted to a sketch of the proof that $\mathfrak{b}$ is continuous. Some related questions on splitting of inverse images of conjugacy classes in the universal covering of G were also discussed. These Theorems were suggested by results of Wood, Milnor (cf. C.M.H. 46(1971) 257) for $G = SL_2(\mathbb{R})$ and Milnor for $G = Sp_{2n}(\mathbb{R})$ (unpublished).

A. Helversen - Pasotto: On the Schur index of representations of $GL(n,F_q)$

In a recent paper Z. Ohmori und T. Yamada proved, by the methods of finite group theory, that all irreducible complex characters of $GL(2,F_q)$ and $GL(3,F_q)$ have Schur Index one.
We show that for every integer $n \geqslant 1$, the Gelfand-Graev representation of $GL(n,F_q)$ is rational (i. e. definable over $\mathbb{Q}$). It follows that every irreducible representation of $GL(n,F_q)$ which admits a Whittaker model has Schur index one. The above mentioned results of Ohmori und Yamada follow easily from this. Generalizations of these results have been pointed out to me by P. Cartier and P. Lusztig.

H.-J. Bartels (Göttingen)