MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 10/1977

Zahlentheorie (Diophantische Approximationen)
6.3. bis 12.3.1977

Unter der Leitung von Herrn Professor Dr. Th. Schneider fand in der Woche vom 6.3. bis 12.3.1977 wieder die schon zur Tradition gewordene Tagung über diophantische Approximationen im Mathematischen Forschungsinstitut Oberwolfach statt. Zahlreiche Teilnehmer aus dem In- und Ausland konnten sich in insgesamt 37 Vorträgen über die neuesten Forschungsergebnisse im Gebiet der diophantischen Approximationen, der transzendenten Zahlen, der Geometrie der Zahlen und der Gleichverteilungstheorie informieren. Daneben kam ein reger Gedankenaustausch in zahlreichen persönlichen Gesprächen zustande, was nicht zuletzt auch der angenehmen Oberwolfacher Atmosphäre zu verdanken ist. In seiner 'problem session' stellte Professor Dr. P. Erdös wieder eine ganze Reihe ungelöster Probleme aus der Zahlentheorie vor.

Teilnehmer

Bertrand, D., Palaiseau
Bijlsma, A., Amsterdam
Binder, Ch., Wien
Brownawell, W.D., University Park
Bumby, R.T., New Brunswick
Bundschuh, P., Köln
Burke, J.R. Carbondale
Cijsouw, P.L., Endhoven
Cohn, H., Kopenhagen
Cusick, Th.W., Amherst
Dubois, E., Caen
Erdös, P., Murray Hill
Flicker, Y.Z., Cambridge
Geijsel, J.M., Amsterdam
Helmberg, G., Innsbruck
Hlawka, E., Wien
Loxton, J.H., Cambridge
Masser, D.W., Nottingham
Mendes France, M., Bordeaux
Mignotte, M., Strasbourg
Niederreiter, H., Urbana
Novak, B., Prag
Philipp, W., Urbana
van der Poorten, A.J. Kensington (Aus)

Gefördert durch DFG Deutsche Forschungsgemeinschaft

Reversat, M., Bordeaux
Reyssat, E., Paris
Schlickewei, H.P., Boulder
Schmidt, A., Kopenhagen
Schmitt, P., Wien
Schneider, Th., Freiburg
Schoißengeier, J., Wien
Schwarz, W., Frankfurt
Schweiger, F., Salzburg
Shidlovsky, A.B., Moskau
Stepanov, S.A., Moskau
Tijdeman, R., Leiden
Väänänen, K., Freiburg
Volkmann, B., Stuttgart
Waldschmidt, M., Paris
Wallisser, R., Freiburg
Wirsing, E., Ulm
Wüstholz, G., Freiburg

Vortragsauszüge

Bertrand, D.: Transcendental numbers in ultrametric domains

The aim of this talk is mainly to establish the p-adic analogues of well-known theorems of Schneider concerning the values of Weierstrass elliptic functions with algebraic invariants. They are deduced from a transcendence criterion for p-adic functions satisfying a system of algebraic differential equations and certain functional equations. The size estimates obtained in the case of exponential function have to be sharpened. This is achieved by an extension of a technical lemma due to Baker, Coates and Masser. This criterion also applies to the logarithm functions associated with classical formal groups, and can be generalised to the study of the exponential map on group varieties. In the case of C.M. abelian varieties, a p-adic adaptation of a method of Coates and Lang yields to more precise results.

Bijlsma, A.: On the simultaneous approximation of a, b and a^b

For any fixed natural number $\varkappa$, there exist irrational numbers in the intervall (0,1) such that for infinitely many triples (α,β,γ) of rational numbers

$$\max(|a-\alpha|,|b-\beta|,|a^b-\gamma|) < \exp(-\log^{\varkappa}H),$$

where H denotes the maximum of the heights of α,β and γ.

For $d \in \mathbb{N}$, there exists an effecitively computable constant $C_1 > 0$, depending only on d, with the following property: if $\varepsilon > 0$, $a \in \mathbb{C}-\{0\}$, l is a branch of the logarithm with $l(a) \neq 0$, and either $b \in \mathbb{C}-\mathbb{R}$ or $b \in \mathbb{R}-\mathbb{Q}$ such that the convergents p_n/q_n of the continued fraction expansion of b satisfy

$$\limsup_{n\to\infty} \frac{\log q_{n+1}}{q_n^3 \log^{4+\varepsilon} q_n} < C_1 ,$$

there are only finitely many triples (α,β,γ) of algebraic numbers of degree at most d satisfying

$$\max(|a-\alpha|,|b-\beta|,|a^b-\gamma|) < \exp(-\log^3 H \log^{1+\varepsilon}\log H),$$

where $a^b = \exp(b\, l(a))$ and H denotes the maximum of the heights of α,β and γ.

Binder, Ch.: Remarks to the large sieve

In the 1-dimensional real case the large sieve inequality can be formulated as follows: let $x_1,\ldots,x_R$ be real points with $\delta = \min_{i,j=1,\ldots,R} \|x_i - x_j\|$ where $\|x\| = \min_{n\in\mathbb{Z}} |x-n|$; let $a_1,\ldots,a_N$ be any real or complex numbers. For the function $S(x) = \sum_{n=1}^{N} a_n e^{2\pi i n x}$ the inequality (called 'large sieve') holds:

$$\sum_{r=1}^{R} |S(x_r)|^2 \leq (N + \delta^{-1}) \sum_{n=1}^{N} |a_n|^2.$$

This can be generalized to the s-dimensional case and to locally compact groups in the following way: let R be a locally compact group, second countable and with a measure, let Γ be a normal subgroup, such that $R/\Gamma = F$ is compact. Then there exists an infinite sequence of unitaire irreducible representations $(D^{(i)}(x))_{i\in\mathbb{N}}$ with degrees r_i of F, which can be extended to R. Let $x_1,\ldots,x_K$ be points in R and $A \subset F$, such that the sets $Ax_k \subset F$ are disjoint. Let $F^{(i)}_{i=1,\dots,n}$ be any $r_i x r_i$-matrices. For the function

$S(x) = \sum_{i=1}^{N} D^{(i)}(x) | F^{(i)}$ holds the following inequality:

$$\sum_{k=1}^{K} |S(x_k)|^2 \leq \frac{V(F)}{V(A)} \left(\frac{\|F^{(1)}\|^2}{r_1^4} + (1-\lambda)^{-2} \sum_{i=2}^{N} \frac{\|F^{(i)}\|}{r_i^4} \right)$$

where $\lambda = \max_{i=2,\dots,N} \sup_{x \in A} \|D^{(i)} - \mathfrak{C}\|, \ \lambda < 1.$

The proof of this inequality was given, leaving out some technical details.

Brownawell, W.D.: <u>On the linear independence of some exponential functions</u>

A quantitative several variables version of Borel's theorem on possible linear dependence relations for functions of the form e^f was described in terms of the Nevanlinna theory for functions of several variables developed by Lelong and Gauthier: Let $\lambda(r) \nearrow \infty$ and $O_\lambda = \{$meromorphic f: $T(r,f)=O(\lambda(r))$ a.a.r$\}$. Here 'a.a.r' means for all but a set of finite Lebesgue measure. Let $f_1,\dots,f_m$ be entire functions.

<u>Theorem 1</u>: $e^{f_1},\dots,e^{f_m}$ are linearly dependent over O_λ if and only if $e^{f_i - f_j} \in O_\lambda$, some $1 \leq i < j \leq m$.

<u>Theorem 2</u>: Let $e^{f_1},\dots,e^{f_m}$ be linearly independent over O_λ. Let $g_1,\dots,g_m \in O_\lambda$, none identically zero. Set

$$G = g_1 e^{f_1} + \dots + g_m e^{f_m} .$$

Then $T(r,G) = \max_i T(r,e^{f_i}) + O(\lambda(r)) + \log r + \max_i \log T(r,e^{f_i}))$.

The proof follows the general outlines of Nevanlinna's classical proof of the theorems of Borel and Picard, but using the derivation $Df = z_1 \frac{\partial f}{\partial z_1} + \dots + z_m \frac{\partial f}{\partial z_m}$ instead of ordinary differentiation.

Bumby, R.T.: The Markoff Spectrum

Survey of recent work on: gaps in the spectrum below $\sqrt{12}$; the segment of measure zero, relation with Lagrange spectrum; the beginning of Hall's ray.

Burke, J.R.: Independence of sequences of Gaussian integers mod μ

We consider the distribution properties of sequences (α_n) in Z[i]. The distribution is defined by

$$\lim_{N\to\infty} \frac{A(N,\gamma,\alpha_n)}{N} = \|A(\alpha_n \equiv \gamma)\| = \Phi_\alpha(\gamma)$$ where $A(N,\gamma,\alpha_n)$ = number of indices $1 \leq n \leq N$ such that $\alpha_n \equiv \gamma \pmod{\mu}$. This is then extended to sequences of ordered pairs (α_n, β_n) obtaining

$$\lim \frac{A(N,\xi,\alpha_n,\eta,\beta_n)}{N} = \|A(\alpha_n \equiv \xi, \beta_n \equiv \eta)\|$$ where both congruences must be satisfied. Let (α_n) and (β_n) be sequences in Z[i] such that $\|A(\alpha_n \equiv \xi, \beta_n \equiv \eta)\|$ exists for all $\xi, \eta \in \Lambda$ (Λ a complete residue system mod μ). The sequences are independent mod μ if for all $\xi, \eta \in \Lambda$ $\|A(\alpha_n \equiv \xi, \beta_n \equiv \eta)\| = \|A(\alpha_n \equiv \xi)\| \|A(\beta_n \equiv \eta)\|$.

Among the results we have

1. (α_n) and (β_n) are independent mod μ if and only if for all $\lambda, \nu \in$ Z[i] the sequence $(\lambda\alpha_n + \nu\beta_n)$ has for a distribution

$$\sum_{\substack{\xi,\eta \in \Lambda \\ \lambda\xi + \nu\eta \equiv \gamma \pmod{\mu}}} \|A(\alpha_n \equiv \xi)\| \|A(\beta_n \equiv \eta)\| .$$

2. (α_n) and (β_n) are independent mod μ if and only if for all $\lambda, \nu \in$ Z[i]

$$\lim \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \,\mathrm{Re}((\lambda\alpha_n + \nu\beta_n)/\mu)} = \lim \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \,\mathrm{Re}\,(\lambda\alpha_n/\mu)} \times \lim \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \,\mathrm{Re}\,(\nu\beta_n/\mu)} .$$

Cijsouw, P.L.: Linear dependence relations of logarithms of algebraic numbers

In 1968 Baker published an estimate for the absolute value of a linear form of logarithms of algebraic numbers which was completely explicit and so could be used for numerical applications. Later he obtained from this work a 'low dependence relation' for given linearly dependent logarithms of algebraic numbers. It will be shown, that both results can be improved considerably by means of just the same method, Baker used.

Cohn, H.: Use of geodesics in estimating Markoff numbers

In earlier work (Ann.Math. 1955, Acta Arith. 1971) it was shown that the Markoff triples (m,m',m'') $m>m'>m''>0$, $m^2+m'^2+m''^2 = 3mm'm''$ are given by 1/3 trace of matrices $M(u,v)$, $M(u',v')$, $M(u'',v'')$, $u'>0$, $u''>0$, $v'>0$, $v''>0$, $u'v'' - v'u'' = \pm 1$, $(u,v) = (u',v') + (u'',v'')$, $M(u,v) = \prod_{s=1}^{u} V_2 V_1^{[sv/u]-[(s-1)v/u]}$, $M(0,1) = V_1$, $V_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $V_2 = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$, $V_1x = 1+1/1+1/x$, $V_2x = 2+1/2+1/x$. Markoff forms have the same roots as fixed points θ of these matrices and $|\theta - p/q|$ permits $q^{-2}(9-4/m^2)^{-1/2}$. The proof uses geodesics in the foundamental domain of $[\Gamma,\Gamma]$ where $\Gamma = PSL_2(Z)$. Similarly, we estimate the distribution of Markoff numbers $\{m_N = 1,2,5,13,29,34,89,\ldots\}$ as $N\to\infty$, $\limsup 3m_N/\sqrt{N} \leq \pi(\log(3+\sqrt{8}) \log\frac{1}{2}(3+\sqrt{5})/3)^{1/2} = 2.36247\ldots$ and, 'likely', lim = constant within 1% of the value cited, by peculiarities of the geodesic configuration.

Cusick, Th.W.: Integer multiples of periodic continued fractions

The results of H. Cohn (Acta Arith. 26(1974-75), 129-148) are obtained much more simply. The new idea is to use an old algorithm of A. Châtelet (Bull.Soc.Math. France 40 (1912), 1-25) for computing the partial quotients of the continued fraction for $N\alpha$ (N positive integer, α real), given the partial quotients of the continued fractions for α. The Châtelet algorithm is busically the same as given in a paper of Mendes France (Acta Arith. 23(1973), 207-215) and used by Cohn, but the Châtelet algorithm is in a far much more suitable for the problems at hand.

Flicker, Y.Z.: On p-adic G-functions

Let K be a number field and denote by $|\ |_p$ any p-adic valuation on K with $|p|_p = p^{-1}$. We study algebraic independence properties of p-adic G-functions using the Siegel-Shidlovsky method. A power series $g(z) = \sum_{n=0}^{\infty} a_n z^n$ with $a_i \in K$, for which there exist $c > 1$ and natural numbers $b_0, b_1, \ldots$ such that $b_n a_0, \ldots, b_n a_n$ are algebraic integers and $b_n \le c^n$, $\text{size}(a_n) \le c^n$, is a p-adic G-function, it converges p-adicaly for all z with $|z|_p < c^{-1}$. Consider G-functions satisfying a system of linear differential equations with coefficients in K(z). Now let $g_1(z), \ldots, g_m(z)$ be solutions of this system, which do not satisfy any algebraic equation with degree $\le r$ over K(z). Let $P(X_1, \ldots, X_m)$ be any polynomial with degree $s \le r$, with integer coefficients in K, whose sizes are $\le H$. Put $n = \binom{r+m}{m}$, $v = \binom{r-s+m}{m}$, and suppose $t(1-\frac{v}{n}) < 1$, $t = [K:Q]$.

Theorem: For every $\epsilon > 0$, there exists $\delta > 0$ such that for any natural numbers q, q' with $q < q'$, $(q,q')=1$ and $|q|_p \le q^{-1+\epsilon}$, we have

$$|P(g_1(q/q'), \ldots, g_m(q/q'))|_p \gg H^{-\lambda-\epsilon},$$

where $\lambda = tv/(1-t(1-\frac{v}{u}))$, and the implied constant depends on $g_1, \ldots, g_m, q, q', r, s, \epsilon$ but not on H.
Note that when the polynomial P is actually a linear form, the lower bound is best possible.

Geijsel, C.: Transcendency in fields of characteristic p

Let $\mathbb{F}_q$ be a finite field; $\mathbb{F}_q(X)$ denotes the quotient field of the ring of polynomials $\mathbb{F}_q[X]$. Let Φ denote a complete non-archimedean valued algebraically closed field that contains $\mathbb{F}_q(X)$. The first results on transcendency in Φ can be considered as analogues of well-known classical results (L.I. Wade 1941,1946) Another result is the following:

Let $f: \Phi \to \Phi$ be given by $f(t) := \sum_{k=0}^{\infty} c_k \frac{t^{q^k}}{F_k}$; $c_k \in \mathbb{F}_q$, $c_k \neq 0$, where $F_k = \prod_{i=0}^{k-1} (X^{q^k} - X^{q^i})$. If $\alpha \in \Phi$, $\alpha \neq 0$ is algebraic over $\mathbb{F}_q(X)$, then $f(\alpha)$ is transcendental over $\mathbb{F}_q(X)$.

Helmberg, G.: Über (C,1)-wesentliche Indexfolgen

Eine Teilfolge $(n_j)_{j\in N}$ der natürlichen Zahlen heißt (C,1)-wesentlich, wenn für jede beschränkte (C,1)-summierbare Folge reeller Zahlen $(a_n)_{n\in N}$ auch die Teilfolge $(a_{n_j})_{j\in N}$ zum gleichen Grenzwert (C,1)-summierbar ist. Indexfolgen dieser Art sind für Gleichverteilungsfragen von Interesse: mit einer Folge $(x_n)_{n\in N}$ ist auch die Teilfolge $(x_{n_j})_{j\in N}$ im jeweils zugrunde liegenden Raum gleichverteilt.
Die von H. Rindler 1974 erzielten Resultate über Existenz und Struktur wesentlicher Indexfolgen werden verschärft und ergänzt.

Hlawka, E.: Zur quantitativen Theorie der Gleichverteilung

Es sei $x_1,\dots,x_N$ eine Folge in $E = [0,1)$. Es wird die klassische Diskrepanz

$$D_N = \sup_{J\subset E} \left| \frac{1}{N} \sum_{k=1}^{N} c_J(x_k) - \lambda(J) \right|$$

(J Teilintervall von E, c_J charakteristische Funktion,

$\lambda(J)$ Länge $\beta-\alpha$ von $J = [\alpha,\beta)$) verglichen mit der Polynomdiskrepanz

$$P_N = \sup_{i \in N} \left| \frac{1}{N} \sum_{k=1}^{N} x_k^{\,i} - \frac{1}{i+1} \right|$$

und der L-Diskrepanz

$$P_N^{(L)} = \sup_J \left| \frac{1}{N} \sum_{k=1}^{N} x_k^{\,L} c_J(x_k) - \int_E x^L c_J(x)dx \right|$$

verglichen. Es ist

$$P_N \leq D_N \text{ und } D_N \leq 1/(\log(1/P_N)),\ c/N \leq P_N \quad \text{und}$$

$$P_N^{(L)} \leq D_N\ ,\ D_N \leq (P_N^{(L)})^{\frac{1}{2(L+1)}},\ P_N^{(L)} \geq \frac{c}{N+L}\ .$$

Verallgemeinerungen und Probleme wurden angegeben.

Literatur: Zur quantitativen Theorie der Gleichverteilung, Sitz.ber.d.Österreichischen Akad. d. Wiss., Math. naturw. Kl.II, 184(1975), 356-357.

Loxton, J.H.: Multiplicative relations in number fields

I report on some joint work with A.J. van der Poorten on the computation of the effectively computable constants in the latest form of Baker's inequality for linear forms in logarithms. In the course of this project, we needed the following interesting lemma: If $\alpha_1,\ldots,\alpha_n$ are multiplicatively dependent algebraic numbers, then they necessarily satisfy a multiplicative relation with relatively small exponents. A result of this type is implicit in Baker's work, but a direct proof of the lemma gives a better bound for the exponents. It is also remarkable that this bound depends essentially on the product of the logarithms of the heights of the numbers α_j.

Masser, D.W.: Linear forms associated with elliptic and Abelian functions

We describe some recent progress in estimating linear forms Λ in algebraic points of elliptic or Abelian functions. When Λ can be proved non-zero, I proved some time ago that

$$|\Lambda| > C\, e^{-H^{\varepsilon}}$$

for any $\varepsilon > 0$, where H measures the height of the coefficients in Λ. Coates and Lang improved this to

$$|\Lambda| > C\, e^{-(\log H)^{\kappa}}$$

for 'homogeneous' 'non-singular' forms Λ. My research student M. Anderson has recently proved

$$|\Lambda| > C\, e^{-\log H\,(\log\log H)^{\kappa}}$$

for 'inhomogeneous' forms in algebraic points of elliptic functions, and I have generalized his work to Abelian functions. I have also obtained weaker results for 'singular' forms; they imply, for example, that for any $\varepsilon > 0$

$$|B(\tfrac{1}{5},\tfrac{1}{5}) - \beta| > C\, e^{-(\log H)^{4+\varepsilon}}.$$

Mendes France, M.: <u>Continued fractions with large partial quotients</u>

Let ξ be a real number. Denote its continued fraction by

$$\xi = c_0(\xi) + \frac{1}{/c_1(\xi)} \Big/ + \frac{1}{/c_2(\xi)} \Big/ + \ldots$$

Let $h \geq 1$ be a real number. Define the set

$$\Gamma(\xi,h) = \{n \in N \mid c_n(\xi) \geq h\}.$$

Using an idea of Davenport, we prove the two following results.

<u>Theorem 1</u>: Let ξ be rational. Let p be a prime number. Then

$$\sum_{j=0}^{p-1} \operatorname{card} \Gamma(\xi + \frac{j}{p}, p-1) \geq \frac{1}{2}\, \psi(p\xi)$$

where $\psi(p\xi)$ denotes the length of the continued fraction of $p\xi$.

<u>Theorem 2</u>: Let ξ be an irrational number. There exists an integer $a \in \{0,1,\ldots,p-1\}$ such that the upper density $\delta^{*}(a)$ of $\Gamma(\xi + \frac{a}{p}, p-1)$ verifies

$$\delta^*(a) \geq (4p(1+\alpha\log p))^{-1}, \qquad \alpha = (2\log\frac{1+\sqrt{5}}{2})^{-1} = 1.039\ldots$$

On the other hand, for all a

$$\delta^*(a) < ((p-1)\log 2)^{-1}.$$

Mignotte, M.: Transcendency measures

In Mahler's classification of transcendental numbers Ridout's theorem can be applied to generalize results of Baker. A typical result is the following.

Theorem 1: Let ξ be a real number such that there exist $\varepsilon > 0$, $p_1, p_2, \ldots$, $n_1, n_2, \ldots$ satisfying

$$\left|\xi - \frac{p_h}{2^{n_h}}\right| < 2^{-(1+\varepsilon)n_h}, \quad h = 1,2,\ldots \quad \text{and } \limsup \frac{n_{k+1}}{n_k} < \infty.$$

Then ξ is not a U-number.

A theorem of C. Stewart contains the following corollary:

Theorem 2: If θ is an algebraic irrational, $\theta > 1$, then there exists an effective positive constant such that

$$P([\theta^n]) \geq C \log n, \text{ when } \theta^n \notin \mathbb{N},$$

where P(m) is the greatest prime factor of m.

C. Stewart, M. Waldschmidt and M. Mignotte proved the following result, which can be compared with a result of Blansky and Montgomery.

Theorem 3: There exists an effective constant D_0 such that if β is an algebraic integer of degree $D \geq D_0$ satisfying

$$|\beta - 1| < \frac{\log D}{D^2} \quad \text{and} \quad \overline{|\beta|} < 1 + \frac{\log D}{5D^2}$$

then β is a root of unity.

The proof uses classical transcendental methods.

Niederreiter, H.: Discrete diophantine approximations and pseudo-random numbers

Let $x_1, x_2, \ldots$ be a sequence of linear congruential pseudo-random numbers with modulus $m \geq 2$ and multiplier a coprime to m. By using a suitable measure for the statistical independence of s-tuples $(x_n, x_{n+1}, \ldots, x_{n+s-1})$ of successive terms, it turns out that the PRN have favorable independence properties if a belongs to the largest exponent mod m and the lattice point $\underline{a} = (1, a, \ldots, a^{s-1})$ is 'badly approximable mod m' in the sense that $\underline{h} \cdot \underline{a} \equiv 0 \pmod{m}$ holds only for those lattice points $\underline{h}$ which are relatively far away from the origin. In a different context, namely that of numerical integration, such lattice points were considered earlier by Korobov and Hlawka. For s=2, the condition on $\underline{a}$ may be rephrased in terms of the continued fraction expansion of a/m. This ties in with recent work on the question whether for each m there exists a reduced fraction a/m for which all partial quotients in the continued fraction expansion are bounded by a universal constant.

Novak, B.: Diophantine approximations and lattice point theory

From the papers of Jarník, Divis and the author follows that the exact order of the lattice remainder term in the theory of lattice points in more dimensional ellipsoids (with arbitrary centers and weights) depends on a special type of diophantine approximation of the parameters involved. A sketch of the proof of Ω-estimates in the case, when quadratic form has integral coefficients, was given. (See, for example, Math. Notes 17(1975), 669-679)

Philipp, W.: Metric theorems on the distribution of lacunary sequences

Let $\{n_k, k \geq 1\}$ be a lacunary sequence of real numbers. Let D_N be the diskrepancy of the sequence $\{\langle n_k \omega\rangle ; k=1,2,\ldots,N\}$ where, as usual $\langle x\rangle$ denotes the fractional part of x. The following result is an extension of a theorem which appeared in Acta Arth. 26(1975).

Theorem 1: For allmost all ω

$$\frac{1}{4} \leq \limsup_{N\to\infty} \frac{N\, D_N(\omega)}{\sqrt{N \log\log N}} \leq c(q).$$

Here $c(q)$ depends on q, where $n_{k+1}/n_k \geq q > 1$.

Let V be the class of functions with period 1, mean value 0 and of variation not exceeding 2. Then theorem 1 and Koksma's inequality imply at once

$$(*)\quad \frac{1}{4} \leq \limsup_{N\to\infty} \frac{\sup_{f\in V} \left|\sum_{k\leq N} f(n_k\omega)\right|}{\sqrt{N \log\log N}} \leq 2\, c(q) \quad \text{a.e.}$$

The following result is joint work with R. Kaufman. For $0 < \alpha \leq 1$ let Λ_α be the class of functions of period 1, mean value 0 and satisfying $|f(x) - f(y)| \leq |x - y|^\alpha$ $(0 \leq x,y \leq 1)$.

Theorem 2: (*) remains valid with V replaced by Λ_α if $\alpha>1/2$, it becomes false for $\alpha<1/2$. The case $\alpha=1/2$ is an open question.

van der Poorten, A.J.: Applications of Gelfond-Baker methods

This is a report on recent work of mine and various collaborators.

(i) Fermat's equation: Professor K. Inkeri and I noticed that recent refinements of the inequalities (complex and l-adic) for linear forms in logarithms imply that in $x^p+y^p = z^p$; $0 < x < y < z$, $(x,y,z) = 1$ one has, writing: $y - x = k_0 l_1^{w_1} \ldots l_s^{w_s}$ (with the l_i primes $< p$; $w_i \geq 0$), $k_0 > (z-x)^{1-\epsilon}$ and $\epsilon < C'(\log p)^3/(p-1)$, with $C' = C(1+l_1+\ldots+l_s)$, where C is an effectively computable constant ($C = 2^{20^s}$ will do).

(ii) Elliptic curves of conductor 11: Following the arguments of an unpublished manuscript of M. Agrawal and J. Coates, I and Dr. D.C. Hunt have checked the inequalities (both complex and 11-adic) that arise from the potential existence of curves of conductor 11. The practical techniques necessary for turning an effectively computable bound to a useful bound, are of some interest. and will be described. Our work verifies Weil's conjecture for $N = 11$; that is, all elliptic curves of conductor 11 are parametrised by modular functions for $\Gamma_0(11)$.

Reversat, M.: Eutaxic sequences

Let (u_n) be a sequence of elements of $(R/Z)^d$ and (ϵ_n) a sequence of positive real numbers. Let x be an element of $(R/Z)^d$. We study the simultaneous nonhomogeneous diophantine problem: to compute the number $\nu(x,N)$ of positive integers $n \leq N$ such that $\|x - u_n\|_d < \frac{1}{2}\epsilon_n$ (where $\|\cdot\|_d$ denotes the usual distance in $(R/Z)^d$). The sequence (u_n) is called eutaxic (resp. strongly-eutaxic) if for all decreasing sequences (ϵ_n) verifying $\Sigma\, \epsilon_n{}^d = +\infty$ and for allmost all $x \in (R/Z)^d$, we have $\nu(x,N) \to \infty$ (resp. $\nu(x,N) \sim \sum_{1\leq n\leq N} \epsilon_n{}^d$). Among other results, we prove that almost all sequences (relatively to the Haar measure of $[(R/Z)^d]^N$) are eutaxic. On the other hand, the sequence $(n\alpha)$ is eutaxic (resp. strongly-eutaxic) if and only if the Markoff-constant $M(\alpha) = \limsup_{n\to\infty}((n\|n\alpha\|_d)^{-1})$ is finite, in other words, the sequence $(n\alpha)$ is eutaxic (resp. strongly-eutaxic) for almost no α.

Reyssat, E.: Lower bounds for linear forms in elliptic constants

Let $\wp$ be a Weierstrass function with algebraic invariants g_2, g_3, and $\{\omega_1,\omega_2\}$ a fundamental pair of periods of $\wp$, η_1,η_2 the quasi-periods of the ζ-function corresponding to ω_1,ω_2. In his thesis, D.W. Masser gave a lower bound for linear forms of the type

$$\Lambda = \alpha_0+\alpha_1\omega_1+\alpha_2\omega_2+\beta_1\eta_1+\beta_2\eta_2+\gamma 2\pi i \qquad (\alpha_i,\beta_i,\gamma \text{ algebraic})$$

when $\alpha_0 \neq 0$. I shall give new results for two particular forms with $\alpha_0=0$. The first is a transcendence measure for ω_1/η_1, that is a lower bound for $|\Lambda|$ when $\alpha_0=\alpha_2=\beta_2=\gamma=0$. The second is a transcendence measure for ω_1/π, or a lower bound for $|\Lambda|$ when $\alpha_0=\beta_1=\alpha_2=\beta_2=0$. In the case of complex multiplication, the above result of Masser gives a sharper transcendence measure for ω_1/π.

Schlickewei, H.P.: P-adic T-numbers do exist

It was for a long time an open problem whether in Mahler's classification of the real numbers (1932) the class of T-numbers is empty or not. In 1968 W.M. Schmidt using a result of E. Wirsing (1961, but only published in 1969) was able to settle this question. Here we consider the same problem for the p-adics. We start with the following definition, which generalizes the corresponding notion introduced by Koksma (1939) for the reals. Given $\xi \in Q_p$ put

$$\omega_n^*(\xi,H) = \min_{\substack{\alpha\in Q_p \\ \alpha \text{ alg.} \\ H(\alpha)\le H \\ \deg(\alpha)\le n}} |\xi - \alpha| \quad \text{and}$$

$$\omega_n^*(\xi) = \overline{\lim_{H\to\infty}} -(\log(\omega_n^*(\xi,H))/(\log H) ,$$

$$\omega^*(\xi) = \overline{\lim_{n\to\infty}}\, \omega_n^*(\xi)/n,$$

finally denote the smallest integer n, such that $\omega_n^*(\xi) = \infty$, if such an n exists, by $\mu^*(\xi)$. Call a number $\xi \in Q_p$ a T^*-number if $\omega^*(\xi) = \infty$ and $\mu^*(\xi) = \infty$. Using the author's p-adic generalization of the Thue-Siegel-Roth-Schmidt theorem we show

Theorem: Let $A_1, A_2, \ldots$ be a sequence of real numbers satisfying the conditions $A_1>8$ and $A_i \ge 3i^2 A_{i-1}$ for $i>1$ then there exists a number $\xi \in Q_p$ with $\omega_i^*(\xi) = A_i$.

Schmidt, A.: Diophantine approximation of complex numbers

Let $\mathbb{Q}(\sqrt{D})$ be an imaginary quadratic number field with maximal order $\mathbb{Z}_D$. For $\xi \in \mathbb{C} - \mathbb{Q}(\sqrt{D})$ the approximation constant is $c_D(\xi) = \limsup(|q||q\xi-p|)^{-1}$, where the limsup is taken over all $(p,q) \in \mathbb{Z}_D \times (\mathbb{Z}_D - \{0\})$. The Hurwitz spectrum is $H_D = \{c_D(\xi) \mid \xi \in \mathbb{C} - \mathbb{Q}(\sqrt{D})\}$. The lecture will be a report on recent investigations on the part of H_D lying below its smallest limit point, and comprising in particular a complete description in the case $D = -3, -4, -11$.

Schmitt, P.: Lineare Gleichverteilung

Wir betrachten eine Folge $\{x_n\} \subset M \subset \mathbb{R}^3$. Sie heißt linear gleichverteilt (mod U), wenn $\lim\limits_{n\to\infty} \frac{S(n,V+p)}{A(n,V+p)} = p$ f.a. $V+p \subset M$ gilt, wo $V \subset U$ (ein System symmetrischer Umgebungen) erfüllt ist. Dabei ist $A(n,V+p) = \sum\limits_{i\le n} C_{V+p}(x_i)$ die Anzahl der Folgenglieder $x_i \in V+p$, $i \le n$, und $S(n,V+p) = \sum\limits_{i\le n} x_i C_{V+p}(x_i)$ die Summe dieser Folgenelemente. Ist die Folge gleichverteilt, so ist $\{x_n\}$ linear gleichverteilt, ist $M = \bar{I}^3$, der abgeschlossene Einheitswürfel, und U die Menge der symmetrischen abgeschlossenen achsenparallelen Würfel, so gilt: linear gleichverteilt $\Rightarrow$ gleichverteilt; ist $M=\mathbb{R}^3$, U wie oben, so gilt: linear gleichverteilt $\Rightarrow$ relativ gleichverteilt. Die Definition kann auch in Banachräumen verwendet werden, wobei lim = schwache Konvergenz. Ferner wird der Ausdruck betrachtet:

$$\lim_{n\to\infty} \frac{\sum\limits_{i\le n} x_i f(x_i-p)}{\sum\limits_{i\le n} f(x_i-p)} = p,$$

wo $f(x_i) = f(-x_i)$ gilt. Außerdem wird der Zusammenhang von linearer Gleichverteilung mit Gleichverteilung bzw. relativer Gleichverteilung bei anderer Wahl von U diskutiert.

Schoißengeier, J.: Über die Diskrepanz gewisser Folgen mod 1 mit Hilfe diophantischer Approximationen

Wir nennen eine Folge $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^N$ Hartman-gleichverteilt, wenn für jedes $\alpha\in\mathbb{R}$, $\alpha\neq 0$, $(\alpha x_n)_{n\in\mathbb{N}}$ gleichverteilt mod 1 ist. Es gilt dann nach einem Ergebnis von Niederreiter und Schoißengeier, daß für jede Hartman-gleichverteilte Folge, für jede fastperiodische Funktion g und für jedes $\alpha \in \mathbb{R}$, so daß $\alpha\pi$ keine rationale Linearkombination von Fourierexponenten von g ist $(\alpha x_n + g(x_n))_{n\in\mathbb{N}}$ gleichverteilt mod 1 ist. Ist $f: \mathbb{R}^+ \to \mathbb{R}$ stetig differenzierbar, $\lim_{x\to\infty} f(x)/x = \lim_{x\to\infty} 1/(xf'(x)) = 0$ und f' monoton fallend, so ist $(f(n))_{n\in\mathbb{N}}$ Hartman-gleichverteilt. Sind die Fourierexponenten ω_i von g alle algebraisch, $g = \sum_{i\leq k} h_i$, h_i stetig differenzierbar mit Periode $2\pi/\omega_i$, $\alpha\pi$ eine S-Zahl, eine T-Zahl oder algebraisch von einem Grade $> \prod_{1\leq i\leq k} \text{grad}\,\omega_i$, so wird die Diskrepanz der Folge $(\alpha f(n) + g(f(n)))_{n\in\mathbb{N}}$ explizit abgeschätzt nach oben.

Schwarz, W.: Liouville-Numbers and the Theorem of Baire

Let $\mathfrak{X}$ be a complete metric space, $I \subset \mathfrak{X}$ an open, nonvoid subset. Assume that the functions $F_\rho: I \to \mathfrak{X}$ are continuous and open, $\rho = 1,2,\ldots$. If $\mathfrak{G}$ is any dense G_δ-set in $\mathfrak{X}$, there is a G_δ-set $\mathfrak{D} \subset I \cap \mathfrak{G}$, dense in I, such that for any $x \in \mathfrak{D}$ all values $F_1(x), F_2(x),\ldots$ are in $\mathfrak{G}$.

Corollary: There is a dense G_δ-set $\mathfrak{D}$ of (real) Liouville-numbers, such that for $x \in \mathfrak{D}$ all values $F_1(x), F_2(x),\ldots$ are Liouville-numbers.

It is conjectured, that the corollary is true with 'Liouville-numbers' replaced by 'U-numbers of degree n'.

Schweiger, F.: Zahlentheoretische Transformationen mit σ-endlichem invarianten Maß

Die Darstellungen reeller Zahlen durch verschiedene Algorithmen (Dezimalbrüche, Kettenbrüche ...) waren Anlaß, gewisse Transformationen des Einheitsintervalles in sich zu betrachten ($x \mapsto 10x \bmod 1$, $x \mapsto x^{-1} \bmod 1$ ...). Von grundsätzlichem Interesse sind dabei die Fragen, ob diese Transformationen ergodisch sind und ein endliches invariantes Maß besitzen. In letzter Zeit sind nun Verallgemeinerungen dieser Transformationen untersucht worden, die zumeist kein endliches, wohl aber ein σ-endliches Maß besitzen und bei denen auch die Frage der Ergodizität neue Überlegungen erfordern, z.B.

$$x \mapsto \frac{x}{1-x} \bmod 1, \quad x \mapsto x - \frac{1}{x}, \quad x \mapsto x + \sum_{i=1}^{n} \frac{p_i}{\eta_i - x}.$$

Shidlovsky, A.B.: The arithmetic properties of the values of some classes of analytic functions

In the report it will be told of the basic results on the linear independence, transcendence and algebraic independence of the values of E-functions, G-functions and some other classes of analytic functions, about estimates of moduli of linear forms and polynomials with integer coefficients of the values of such functions, including a series of the latest investigations in this direction. Problems of generalizing of the known results to wider classes of functions and wider sets of their arguments will be formulated and a series of unsolved problems which are of interest for the development of the theory of transcendental numbers will be given.

Stepanov, S.A.: A lower bound for some characters sums

Let p be a prime number and N any natural number not exceeding p. We show the existence of square-free with respect to mod p polynomials f(x) and g(x) with integer coefficients of degree $2n > \max(\frac{2(n+1)\log 2}{\log p}, 2)$ such that

$$\sum_{x=1}^{N} \left(\frac{f(x)}{p}\right) = N \quad \text{and} \quad \sum_{x=1}^{N} \left(\frac{g(x)}{p}\right) = -N.$$

Stewart, L.: Primitive divisors of Lucas and Lehmer numbers

Let A and B be non-zero integers of an algebraic number field K. A prime ideal $\mathfrak{P}$ of K is called a primitive divisor of $A^n - B^n$ if $\mathfrak{P} \mid [A^n-B^n]$ and $\mathfrak{P} \nmid [A^m-B^m]$ for $0 < m < n$, here [x] denotes the principal ideal generated by x in K. Schinzel proved that if $([A],[B]) = 1$ and A/B is not a root of unity then A^n-B^n has a primitive divisor for all $n > n_0(d)$, where d is the degree of A/B over $\mathbb{Q}$ and $n_0(d)$ is effectively computable. We prove that $n_0(d)$ may be taken to be the maximum of $2(2^d-1)$ and $e^{452 d^{65}}$. For the case of Lucas and Lehmer sequences we are able to improve this result. We prove that there are only finitely many Lucas and Lehmer sequences whose n-th term, $n > 6$, $n \neq 8$, 10 or 12, does not possess a primitive divisor and these sequences may be explicitly determined. This result is best possible for Lehmer sequences.

Tijdeman, R.: Uniform distribution in finite sets

There are n states, $C_1,\ldots,C_n$, which form a union, and have to select a chairman every year. Let λ_{ij} denote the weight of state C_i in year j, where $\sum_{1\le i\le n} \lambda_{ij} = 1$ for $j=1,2,\ldots$; $\lambda_{ij} \geq 0$. Let A_{iJ} denote the number of chairmen from state C_i after J years. In order to avoid unnecessary large discrepancies (and hence troubles) the first problem was to compute

$$M_n = \sup_{\{\lambda_{ij}\}} \inf_{\left\{\substack{\text{possible}\\ \text{choices of}\\ \text{chairmen}}\right\}} \sup_{i,J} \left|A_{iJ} - \sum_{1\le j\le J} \lambda_{ij}\right|.$$

This problem was posed by H. Niederreiter and solved by H.G. Meijer and R. Tijdeman. Using Hall's theorem on distinct representatives they proved that $M_n = 1 - 1/(2(n-1))$. In the lecture a simple algorithm is given which provides at the same time a simpler proof of the value of M_n. Furthermore, M_n is computed and algorithms are given in cases that certain conditions are imposed upon the weights λ_{ij}.

Väänänen, K.: On lower bounds for polynomials in the values of E-functions

Let $P(x,x_1,\dots,x_s) \not\equiv 0$ be a polynomial with coefficients in $\mathbb{Z}$, and let θ be an algebraic number satisfying certain conditions. We obtain a lower bound in terms of the heights of P and θ for the absolute value of $P(\theta,f_1(\theta),\dots,f_s(\theta))$, where $f_1(z),\dots,f_s(z)$ are certain Siegel E-functions. We also estimate simultaneously the absolute value of $\alpha-\theta$, $\beta-\gamma$, $P(\alpha,f_1(\beta),\dots,f_s(\beta))$, where α,β are given complex numbers and θ,γ are algebraic numbers satisfying certain conditions. We consider separately the special case $s = 1$, $f_1(z) = e^z$, and obtain essentially better bounds than in the general case. All these results are obtained by using the Siegel-Shidlovsky method.

Waldschmidt, M.: Linear forms in two logarithms and Schneider's method (following a joint work with M. Mignotte)

We consider an homogeneous linear form in two logarithms of algebraic numbers with algebraic coefficients: $\beta_1 \log\alpha_1 + \beta_2 \log\alpha_2$. The first lower bound for such a linear form was obtained by Gelfond in 1935. Baker generalized Gelfond's method to obtain a result concerning more general linear forms. This result of Baker had such deep consequences that a lot of papers were written on this subject; these papers have introduced very important improvements of the original method. But, up to now, the inductive argument, which is an essential characteristic of Baker's method does not enable a very precise dependence on the degree. To obtain such an estimate, we use Schneider's method, which, as far as we know, was never used in this context; this means that no derivative is involved in our proof. However we add also some of the above mentioned ideas which were introduced in connection with Baker's method. We apply our lower bound to the simultaneous approximation of numbers; we give an explizite dependence on the degree in the theorem of Franklin and Schneider.

Waldschmidt, M. and Brownawell, W.D.: A report on the paper of G.V. Choodnovsky, '<u>Arithmetical properties of values of analytical functions</u>'.

Let $f(\underline{z})$ be a meromorphic function of strict order $\leq \rho$ in $\mathbb{C}^n$. We generalize theorems of Strauss, Schneider and Waldschmidt on algebraic points of f. Namely we prove:
<u>Theorem</u>: Let $S \subset \overline{\mathbb{Q}}^n$ and for any $w \in S$ and any $k \in \mathbb{N}^n$ every derivative $\partial^k f(w)$ belongs to $\mathbb{Z}[i]$. Then S is contained in an algebraic hypersurface of degree $\leq \rho n + 2n_1$ where $n_1 = n$ for $n \geq 2$ and $n_1 = 0$ for $n = 1$. Analogous results take place for $\mathbb{Z}[i]$ changed by any algebraic field K if some assumptions on denominators and heights of $\partial^k f(w)$ are imposed.
Known theorems on algebraic independence and measure of transcendence of values of E- and G-functions in algebraic points are derived from Gelfond-Schneider's method via singularities. Moreover, some results on algebraic independence of values of functions satisfying linear differential equations in $\mathbb{C}^n$ are also proved. One example connected with elliptic curves and Gauss-Manin connection is the following. For any $k \neq 0$, $|k| < 1$ at least two among the numbers k, $F(1/2,1/2,1;k^2)$, $F'(1/2,1/2,1;k^2)$ are algebraically independent.

Wüstholz, G.: Linearformen in Logarithmen von U-Zahlen

Es seien $a_1, \ldots, a_n$ komplexe Zahlen, deren Logarithmen linear unabhängig über den rationalen Zahlen seien. Wenn die a_i einer gewissen Klasse von U-Zahlen angehören, so kann der folgende Satz bewiesen werden:

Satz : Es seien $x_1, \ldots, x_n$ ganze rationale Zahlen, welche nicht alle gleich 0 seien und deren Absolutbeträge höchstens gleich X seien. Weiter seien $\alpha_1, \ldots, \alpha_n$ algebraische Zahlen eines Grades $\leq d$ und mit Höhen $\leq A$ $(A \geq 4)$. Dann gilt bei beliebigem $\varepsilon > 0$

$$\max\left(\max_{1\leq i\leq n}(|a_i-\alpha_i|), |x_1 \log a_1 + \ldots + x_n \log a_n|\right) > e^{-\log X} (\log H)^{n(n-1)+\varepsilon},$$

falls $A > A_0(n, d, \varepsilon, \log a_i)$.

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