MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 16|1977

Definability in Set Theory

17.4. bis 23.4.1977

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Dank eines günstigen Zusammentreffens mit einer anderen Tagung konnten für diese Tagung eine Reihe von Teilnehmern aus dem Ausland gewonnen werden, die einerseits einen guten Überblick über die neuesten Ergebnisse in der höheren Mengenlehre gegeben haben und andererseits anschließend über ihre eigenen Forschungen vortrugen. Es zeigte sich dabei, daß, verglichen mit den Vereinigten Staaten, wir in Europa erhebliche Kenntnislücken haben. Das zentrale Thema der Tagung betraf den Zusammenhang zwischen dem möglichen Reichtum an Teilmengen des Kontinuums und möglichen Unendlichkeitsaxiomen. Dieser Zusammenhang läßt sich formal fassen durch Axiome aus der deskriptiven Mengenlehre, die über ZFC (Zermelo-Fraenkel mit Auswahl) hinausgehen und durch Axiome über große Kardinalzahlen. Dieser Zusammenhang ist deshalb wichtig, weil das bevorzugte Modell der Mengenlehre, nämlich die kumulative von Neumann'sche Hierarchie von zwei Parametern abhängt: der Potenzmenge und der Klasse der Ordinalzahlen.

Obwohl im Verlauf der Tagung keine Freizeit übrig blieb, ja sogar drei Abendsitzungen eingeschaltet wurden, zeigte sich keine Spur von Ermüdung; wir waren alle bis zur letzten Stunde erfüllt von den Aufgaben, die sich uns stellten. Es ist geplant, daß über diese Tagung Lecture Notes beim Springer Verlag erscheinen sollen.

Gert H. Müller

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Vortragsauszüge

J. BARWISE, Global Inductive Definability.

With Y.N. Moschovakis, we study inductive definitions $I_{\varphi}^{\infty}(\mathfrak{M})$ over $\mathfrak{M} \in \mathfrak{K}$, where $\mathfrak{K}$ is a class of structures PC_{Δ} - Definable in $L_{\omega_1\omega}$. We obtain ordinal bounds on lengths of such definitions by assuming that $\|\varphi\|^{\mathfrak{M}} < \mathfrak{K}(\mathfrak{M})$ for all $\mathfrak{M} \in \mathfrak{K}$. We apply these results in various parts of logic and mathematics. For example, we obtain Sack's effective bound on Morley - Shelah ranks as an immediate corollary.

F. DRAKE, Higher-Type Measurables from Δ_2^1 - Determinacy.

Solovay has shown that if we assume that all Δ_2^1 sets of reals are determined, then there is an inner model with a measurable cardinal. This is generalized to obtain an inner model with a measurable cardinal which is the limit of measurable cardinals.

U. FELGNER, Forcing and Constructibility for Ackermann's Set Theory.

We discuss the following problem: is it possible to express in Ackermann's system of axioms the notion of constructibility by a single formula such that this formular defines an inner model of A^* + "every class is constructible"? Here A^* = Ackermann's four axioms + axiom of foundation relativized to V. Solution: define the constructible levels in a slightly different way by incorporating into the definition of L_β the existence of a satisfaction-function. Let $v = L_\beta^*$ be the resulting formula. Put $\alpha = \{\nu \in V \mid Ord(\nu)\}$ and call an ordinal γ correct iff $\gamma < \alpha$ or if L_γ^* exists and every β with $\alpha < \beta \leq \gamma$ has the property that α is parametrically definable in $(L_\beta^*, \in)$. Then $\Lambda = \{v \mid \exists w . \exists \gamma \ (v \in w = L_\gamma^* \ , \gamma \text{ is correct})\}$ is an inner model of A^* + "all classes are constructible".

Forcing in Ackermann's system A^* is available - but in a generic extension of a model $\mathfrak{M}$ of A^* not all of $\mathfrak{M}$ is extended but only a sufficiently large initial segment of $\mathfrak{M}$. Forcing with classes is not always possible! These are results recently obtained by C. Alcor (PhD-thesis, Tübingen 1977).

H. FRIEDMAN, Π^1_2 and Π^1_3 Theorems Respectively Needing ω_1 Ranks and Morse - Kelley Class Theory.

For functions F, a strong point is an x with $F(x) \in x$. Let $HC_0 = HF$, $HC_\lambda = \bigcup\{HC_\alpha \mid \alpha < \lambda\}$, $HC_{\alpha+1}$ = the set of all countable subsets of HC_α. $HC = HC_{\omega_1}$. Consider the following sentences.

1. For all n, any (finitely) Borel $F|HC_{n+1} \to HC_n$ has a strong point.
2. Any (finitely) Borel $F|HC_{\alpha+1} \to HC_\alpha$ has a strong point, (for all countable α).
3. Any Borel $F|HC \to HC$ such that for $x \in HC_{\alpha+1}$, $F(x)$ is a function with $Dom(F(x)) \in HC_\alpha$ and $Rng(F(x)) \subset HC_\alpha$, has a strong point.

Statement 1 is a Π^1_2 sentence provable in Zermelo set theory but not in the theory of types.
Statement 2 is a Π^1_2 sentence provable in Zermelo set theory + "there is a rank on an uncountable ordinal", but not provable in, say, Zermelo set theory + "there is a rank on a recursively inaccessible ordinal".
Statement 3 is provable in Morse - Kelley class theory, but not provable in ZF. It is Π^1_3.
There are equivalent formulations involving countable trees which may be more mathematically natural (especially in the case of 1).If the statements are arithmetized, they basically maintain there metamathematical properties and become Σ_1, Σ_1, Σ^1_2 respectively.

W. GUZICKI, Second Order Arithmetic with $2^\omega \leq \omega_n$.

By Platek's theorem there is no sentence φ of second order arithmetic such that ZFC + $\varphi \vdash 2^\omega > \omega_1$. On the contrary, we show sentences φ_n such that ZFC + $\varphi_n \vdash 2^\omega \leq \omega_{n+1}$ but ZFC + $\varphi_n \nvdash 2^\omega \leq \omega_n$. We show that such sentences cannot be Σ^1_3, however our examples are far from being that.

The reason that Platek's theorem works can (by virtue of the existence of forcing argument proof) be stated as that there exists a new method of counting reals. This however does not affect the continuum itself, which can still be big. In order to convince ourselves that from the point of view of continuum it is big, we construct interpretations of set theory in the extensions of the second order arithmetic. In some cases it is possible to construct interpretations such that the continuum of the interpretation is isomorphic to the universe of the arithmetic and which contains an uncountable set. By a general fact, the continuum of the interpretation is a proper fact, thus in a sense violating the CH in the arithmetic.

P. HAJEK, Some Remarks on Degrees of Constructibility.

I. If ω_1^L is countable then there is a strictly ascending countable sequence of degrees of constructibility of reals having a minimal upper bound, which is a degree of a real.

II. If $0^\#$ exists then there are Δ^1_3 reals a_n $(n \in \omega)$ and a Δ^1_3 real a such that $\{dg_c(a_n) | n\}$ is strictly ascending and $dg_c(a)$ is its minimal upper bound.

III. Moreover, if $0^\#$ exists then: there are Δ^1_3 Cohen, Solovay, Sacks reals; there is a pair of Δ^1_3 reals whose degrees have no greatest lower bound; there is a sequence $\{a_n \mid n \in \omega\}$ of Δ^1_3 reals such that $\{dg_c(a_n) \mid n \in \omega\}$ is strictly ascending and has no least upper bound; for each finite lattice L, there is an initial segment of degrees of constructibility isomorphic to L and consisting only of degrees of Δ^1_3 reals.

L. HARRINGTON, The Prewellordering Property for Π^1_3.

Definition: PWO(Π^1_3) means: there is a function f: $\mathfrak{U} \to$ Ord (where $\mathfrak{U}$ = universal Π^1_3 set of reals) such that for all a in $\mathfrak{U}$ $\{ b \mid f(b) \leq f(a) \}$ is Δ^1_3 in a, uniformly in a.

PWO(Π^1_3) is one of the many consequences of PD (projective determinacy). It is our belief that most of the structural properties of the projective hierarchy, which follow from the large cardinal property PD, are not themselves large cardinal properties. One particular instances of this is verified by the following Theorem: There is a model of ZFC (assuming that ZFC has a model) in which PWO(Π^1_3) holds.

H.R. JERVELL, Constructive Universes.

The talk is divided into four parts.

1. A survey of the proof-theoretic strength of various constructive systems.
2. An introduction to P. Martin-Löf's theory of types showing how one can develop parts of mathematics within the system.
3. The constructions of subsystems of models in Martin-Löf's system are discussed.
4. Some open questions related to the strength of Martin-Löf's system are settled.

A. KANAMORI, On Vopenka's and Related Principles.

Analogies to Baumgartner's n-subtle and n-ineffable cardinals are developed in context of strong principles of infinity: A sequence $(M_\xi \mid \xi<\kappa)$ of the same similarity type is called natural iff each $M_\xi = (V_{f(\xi)}, \in, A_\xi, \{\xi\})$ and $\xi<\xi^*$ implies

$\xi < f(\xi) \leq f(\xi^*) < \varkappa$. If $X \subseteq \varkappa$, X is Vopenka -n-subtle (Vopenka-n-ineffable) iff for any natural sequence, there is a $Y \subseteq X$, so that $|Y| = n+1$ (Y is stationary in $\varkappa$) and given $\alpha_o < \ldots < \alpha_n$ in Y , there is an elementary embedding j of one model in the sequence into another, so that α_o is the first ordinal moved and $j(\alpha_i) = \alpha_{i+1}$.

Typical results: (a) If $\varkappa$ is Vopenka-n-subtle, then $\varkappa$ is n-subtle. (b) Those $X \subseteq \varkappa$ so that $\varkappa$-X is not Vopenka-n-subtle form a normal filter containing $\{\alpha<\varkappa \mid \alpha$ is (n-1)-huge$\}$ when n>1, and $\{\alpha<\varkappa \mid V_\varkappa \models \alpha$ is extendible$\}$ when n = 1. (c) Statements analogous to (a) and (b) hold for Vopenka-n-ineffable. (d) If $\varkappa$ is n-huge, then $\varkappa$ is Vopenka-n-ineffable. (e) $\varkappa$ is Vopenka-n-ineffable iff the Π^1_2-indescribable filter and Vopenka-n-subtle filter over $\varkappa$ are proper and coherent.

A.S. KECHRIS, Souslin Representations of Projective Sets and Higher Level Analogs of L.

An inner model M of ZFC is called a 2n-level analog of L if

A) $M \models$ there is a Δ^1_{2n}-good wellordering of ω^ω

B) 1) $M \cap \omega^\omega \in \Sigma^1_{2n}$

2) M is a Souslin basis for Σ^1_{2n},

where M is a Souslin basis for Σ^1_m if every Σ^1_m set A can be written as $A = p[T] = \{\alpha \in \omega^\omega \mid \exists f \in \lambda^\omega\ (\alpha,f) \in [T]\}$, where T is a tree on some $\omega \times \lambda$ and $T \in M$.

Assuming PD let $\mathfrak{P}_{2n-1}$ be a complete Π^1_{2n-1} set of reals, $\{\sigma_m\}_{m\in\omega}$ a Π^1_{2n-1} - scale on $\mathfrak{P}_{2n-1}$ and let

$$T_{2n-1} = \{((\alpha(o)\ldots\alpha(k)),(\sigma_o(\alpha)\ldots\sigma_k(\alpha))) \mid \alpha\in\mathfrak{P}_{2k-1}\}$$

be its associated tree (so that $\mathfrak{P}_{2n-1} = p[T_{2n-1}]$). The following proves a conjecture of Moschovakis.

Theorem (HPD). For each $n \geq 1$, $L[T_{2n-1}]$ is a 2n-level analog of L. In particular, $L[T_{2n-1}] \cap \omega^\omega$ is independent of the tree T_{2n-1}.

Here HPD is the hypothesis that every hyperprojective set of reals is determined. The above theorem was proved by Moschovakis for n = 1 (in which case $L[T_{2n-1}] = L$), by Kechris-Martin for n = 2 and by Harrington-Kechris in general. Open problems: 1) Is $L[T_{2n-1}]$ independent of T_{2n-1} for $n \geq 2$? 2) What is an appropriate indiscernibility theory for these higher level analogs of L?

A. LOUVEAU, *-Games.

The games G^* are particular asymmetric games of perfect information introduced by M. Paris. We study the *-games on ω and prove the following results:

(1) For every analytic subset A of ω^ω, the game $G^*_\omega(A)$ is determined.

(2) The statement "For every Π^1_1 subset A, $G^*_\omega(A)$ is determined" is equiconsistent with the existence of an inaccessible cardinal.

(3) In Solovay's model, every game $G^*_\omega(A)$ is determined.

We also study more general asymmetric games and prove for them analogous results, which lead to regularity properties for subsets of ω^ω, related to σ-ideals generated by closed sets.

W. MAASS, Methods of Set Theory in α- and β- Recursion Theory.

Some priority arguments from ordinary recursion theory can be carried over to α-recursion theory by using in addition results about the fine structure of L. For β-recursion theory (β any limit ordinal) more facts about L are used (e.g. Jensen's $\diamondsuit$) and many new problems arise. Theorem: If β is inadmissible and $L_\beta \models [\beta^*$ regular] then incomparable β-recursive β-degrees exist.

M. MAGIDOR, Saturated Ideals and Σ^1_4 Sets.

Assume a measurable cardinal exists.

Theorem (ω_1 carries an ω_2 saturated ideal).

(A) Every Σ^1_3 set is either countable or contains a perfect subset.

(B) CH → Every Σ^1_3 set is Lebesgue measurable and has the Baire property.

ω_1 is ω_2 almost supercompact if $P_{\omega_1}(\omega_2)$ carries an ω_3 saturated normal ideal.

Theorem (ω_1 carries an ω_2 saturated ideal and it is ω_2 almost supercompact + CH).

(A) Every Σ^1_4 set is either countable or contains a perfect subset and it is Lebesgue measurable and has the Baire property.

(B) Every Σ^1_3 set is completely Ramsey.

W. MAREK, On a Class of Models of n^{th} Order Arithmetic.

In the talk we introduce a class of models of the n^{th} order arithmetic $(n \geq 3)$ which is wider than the class of β-models (models absolute for wellordering) on n^{th} order arithmetic investigated by Zbierski in his thesis.

The models now considered are β-models up to the objects of the k^{th} order. Basic properties of such models are considered, but the main result is the following: For every n, for different k those classes are different. In the course of proof infinitary methods (admissible sets) are used.

The lecture reported common work of Zbierski and undersigned.

A.R.D. MATHIAS, $0^{\#}$ and the Existence of p-Points.

A p-filter is a filter F on ω, containing all cofinite sets, such that given $X_i \in F$ $(i<\omega)$, $\exists Y \in F\ \forall i\ (Y \setminus X_i$ is finite).

A filter is rare if given a partition Π of ω into finite set s_i $\exists X\in F\ \forall i\ |X\cap s_i| \leq 1$. A filter is feeble if there is a strictly monotonic $f|\omega \to \omega$ such that

$$\forall X\in F\ \exists j<\omega\ \forall i>j\ X \cap [f(i),f(i+1)) \neq \phi.$$

A p-point is a p-filter that is also an ultrafilter.

The following are known: using CH or MA, p-points, rare filters, and rare p-points may be constructed. (Kunen) It is consistent with ZFC that no ultrafilter is both rare and a p-point. (Jalali-Naini; Talagrand) F is feeble iff, viewed as a subset of 2^ω, it is meagre. Thus feebleness is a measure of the smallness of filters. Feeble p-filters exist - eg. the Frechet filter.

Let $\varkappa$ be the least cardinal of a family $\mathfrak{D} \subseteq {}^\omega\omega$ such that $\forall f\in {}^\omega\omega\ \exists g\in\mathfrak{D}\ \forall i\ f(i) \leq g(i)$. (Ketonen) If $\varkappa = 2^\omega$, there is a p-point. Using Ketonen's arguments and Jensen's covering lemma, the following was proved. Theorem. If $\varkappa<\aleph_\omega$ or if $0^\#$ does not exist, there is a p-filter which is not feeble. If $2^\omega = \aleph_2$, either there is a p-point or there is a rare filter.

K. McALOON, A Sketch of Recent Work of Kirby and Paris.

Let M vary over countable models of Peano Arithmetik (PA). Let $\mathfrak{C}$ be a function on these models such that $\mathfrak{C}(M)$ is a collection of cuts in M (i.e. substructures of M of which M is an end extension). An indicator for $\mathfrak{C}$ is a function $K(x,y) = \mathfrak{z}$ definable in PA such that for all M, for all $a<b\in M$, $K(a,b)$ is non-standard $\Leftrightarrow$ there exists $I \in \mathfrak{C}(M)$ s.t. $a\in I$, $b\notin I$. Define a finite set of integers X to be 0-dense iff $|X| \geq \min X$; define X to be n+1-dense iff $\forall f|X^{[\mathfrak{z}]} \to 2$ there is $Y\subseteq X$, Y n-dense, Y homogeneous for f. Kirby and Paris have proved that "$[x,y]$ is $\mathfrak{z}$-dense" is an indicator for cuts satisfying PA; furthermore, they have shown there are models of PA satisfying $\exists x\ \forall y\ (K(x,y) \leq x)$ but that the standard integers satisfy $\forall z\ \forall x\ \exists y\ (K(x,y) \geq z)$. The statement

$\forall x \exists y\ (K(x,y) \geq x)$ can be shown (by analyzing their work) to be equivalent in PA to "every Σ_1^0 - sentence provable in PA is true". Various iterations are possible and their techniques can be made to yield various results on submodels of models of PA.

K. PRIKRY, Some Uses and Aspects of Finitely Additive Measures.

We discuss some applications of measures (finitely addit.) and ultrafilters over the set of nat. nos in second order arithmetic. We also consider the assumption of the existence of a non-trivial measure over the set of natural nos as a form of the axiom of choice, no further choice being assumed.

Applications to second order arithmetic.

Thm.(Glazer). There is an ultrafilter $\mathfrak{J} \in \beta\omega-\omega$ such that for all $A \in \mathfrak{J}$, $\{n \mid A-n \in \mathfrak{J}\} \in \mathfrak{J}$. The next theorem is a fairly easy consequence of the above Thm of Glazer: Thm. (Hindman). Suppose that $\omega = I_0 \cup I_1 \cup \ldots \cup I_k$. Then there is an infinite $X = \{x_1 < x_2 < \ldots\}$ such that all finite non-empty sums of distinct elements of X belong to only one of the classes I_j $(j=0,\ldots,k)$. Now define for every set $A \subseteq \omega$,

$$d^*(A) = \lim_{n\to\infty} \sup |A \cap n|/n\ ,\quad d_*(A) = \lim_{n\to\infty} \inf |A\cap n|/n,$$

$d(A) = d^*(A) = d_*(A)$ if $d^*(A) = d_*(A)$. $A-A = \{a-b \mid a,b \in A\}$. We use Hindman's Thm. together with Banach means (finitely additive measures extending d and translation invariant) to proof the following theorem of Fürstenberg, and some stronger results: Thm. Suppose $d^*(A_j) > 0$, $(j \leq k)$. Then $\cap\{A_j - A_j \mid j \leq k\}$ does not have arbitrarily large gaps. Another proof of this (involving only elementary methods) was given by Stewart and Tijdeman. Fürstenberg's proof uses countably additive measures (ergodic theory).

The existence of measures as a weak form of choice.

Thm. (Solovay,Christensen). Let us suppose that there is a finitely additive measure μ over ω which is not countably additive. Then there is a set of reals without Baire property. Definition. $\mathfrak{S} \subseteq P(\omega)$ is a Scott family if for every $X \subseteq \omega$, X infinite, there is an infinite $Y \in \mathfrak{S}$, $Y \subseteq X$. Thm. (Mathias) If a non-trivial finitely additive measure over ω exists (see above Thm. for assumption about μ), then there are two disjoint Scott families. Under the assumption of the preceding theorem we prove: Either there is a Lebesgue non-measurable set, or there is a collection of 2^{ω} pairwise disjoint Scott families.

S.G. SIMPSON, Choice Schemata in Second Order Arithmetic.

Let S be the system of second order arithmetic with full comprehension. Consider the following choice schemata in the language of S.

AC(choice): $\forall n\, \exists X\, \varphi(n,X) \rightarrow \exists Y\, \forall n\, \varphi(n,(Y)_n)$.

DC(dependent choice) $\forall X\, \exists Y\, \varphi(X,Y) \rightarrow \exists Z\, \forall n\, \varphi((Z)_n,(Z)_{n+1})$

where n is a natural number variable and X,Y,Z are set variables. It is well known that S + DC implies AC, and that S implies AC and DC for Σ^1_2 formulae φ. Lévy has constructed a model of set theory in which $\aleph_1 = \aleph^L_\omega$. Hence S does not imply AC for Π^1_2 formulae. We use methods of Jensen and Harrington to construct a model of set theory in which every countable family of nonempty sets has a choice function and there is a CPCA linear ordering of reals which is dense and has no descending sequence. Hence S + AC does not imply DC for Π^1_2 formulae. Mostowski (F.M. 15) has shown how to construct global choice functions for countable models of S + DC. These results, as well as those of Guzicki on parameterless AC and DC (F.M. 93), tend to support the view that S + DC is a more natural system than S + AC.

M. SREBRNY, Singular Cardinals and Analytic Games.

I proved the following theorems.

Theorem 1. ZFC + $(\forall n)(2^{\aleph_n} \leq \aleph_\omega) + (2^{\aleph_\omega} \neq \aleph_{\omega+1}) \vdash \Sigma^1_1$-Determinateness (i.e. every analytic game is determinate).

Theorem 2. ZFC + $(\forall n)(2^{\aleph_n} \leq \aleph_\omega)$ + $(2^{\aleph_\omega} \neq \aleph_{\omega+1})$ $\vdash$ every uncountable Π^1_1 set of reals has a perfect subset.

In the above, we may replace the hypothesis "$(\forall n)(2^{\aleph_n} \leq \aleph_\omega)$ + $(2^{\aleph_\omega} \neq \aleph_{\omega+1})$ by a "negative solution to the singular cardinal problem", where by the singular cardinal problem we mean whether it is true that

$$(\forall \text{ singular } \beta)[(\forall \alpha<\beta)(2^\alpha \leq \beta) \Rightarrow (2^\beta = \beta^+)].$$

Expressed otherwise: ZFC + $\neg\Sigma^1_1$-Determinateness $\vdash$ positive solution to all cases of the singular cardinal problem.

To prove these we relativize the argument of Devlin and Jensen's Marginalia. Our proof uses Silver machines. In particular, we construct Silver $L[a]$-machines with the condensation, finiteness and Skolem properties. Then theorems 1 and 2 follow from the Harrington-Martin result:

$$\Sigma^1_1\text{-Determinateness} \Longleftrightarrow (\forall a \subset \omega)(a^\# \text{ exists}).$$

K. STEFFENS, Matchings.

A survey of recent results of the so-called "Transversal Theory" was given. Some open problems were indicated.

J. STERN, Σ^1_1 Partitions of ${}^\omega\omega$ into Borel Sets of Bounded Rank.

A Σ^1_1 partition of ${}^\omega\omega$ is a partition such that the associated equivalence relation is Σ^1_1 as a subset of $({}^\omega\omega)^2$. We prove the following theorem. Theorem: Assume $\forall\alpha$ ($\alpha^\#$ exists); then any Σ^1_1 partition of ${}^\omega\omega$ into Borel sets of bounded rank has at most $\aleph_0$ classes or admits a perfect set of pairwise inequivalent elements.

[A family of Borel sets is of bounded rank if all the Borel sets of the family are Σ^0_ξ for a fixed countable ordinal ξ, where Σ^0_ξ is defined inductively by Σ^0_1 = open sets, Π^0_ξ = complements of Σ^0_ξ sets, Σ^0_ξ = countable union of sets in $\bigcup\{\Pi^0_\eta \mid \eta<\xi\}$]

The proof relies on a characterization of Σ^1_1 partitions which admit a perfect set of inequivalent elements among those which are not countable. Let p be such a partition and E the associated equivalence relation. A Σ^1_1 subset of ${}^\omega\omega$ is large if it meets uncountably members of p. A large set A splits if there exists large $A_1, A_2 \subseteq A$ such that $\forall\alpha \in A_1 \forall\beta \in A_2\ \alpha \not E \beta$. It splits densely if any large $B \subseteq A$ splits. Thm: E has a perfect set of inequivalent elements if and only if some large Σ^1_1 set splits densely.

P. STEPANEK, <u>Cardinals in the Inner Model HOD</u>.

It is provable in the set theory that $L \subseteq HOD \subseteq V$. Thus for the corresponding classes of cardinals we have

$$(1) \quad \text{Card} \subseteq \text{Card}^{HOD} \subseteq \text{Card}^{L}.$$

Using a new embedding theorem for Boolean algebras and Boolean valued models, it is possible to discuss various possible cases of equalities and inequalities in (1). Actually it is possible to show that all the three classes of cardinals can have arbitrarily large common initial segments plus every combination of equalities and inequalities in (1) is possible, whenever it is compatible with the equalities between L, HOD, V. Everything is done by generic extensions of L of type L[a]. In all models, we can ask HOD to satisfy either V = L or V = HOD or to be "something between L and HOD". One more theorem about decreasing sequences of HOD classes. Theorem: Let $\lambda_0 > \lambda_1 > \ldots > \lambda_n > \varkappa$ be a decreasing sequence of cardinals in L. Let for each $i \leq n$, if $\lambda_i = \mu^+$ then $cf(\mu) > \varkappa$. Let $\varkappa$ be regular. Then there is a generic extension $\mathfrak{M}$ of L such that

(i) $\lambda_i = (\varkappa^+)^{\mathfrak{M}_i}$ for each $i \leq n$

(ii) every constructible cardinal $\mu \leq \varkappa$ or $\mu > \lambda_i$ is a cardinal of $\mathfrak{M}_i$ for each $i \leq n$,

where $\mathfrak{M}_0 = \mathfrak{M}$, $\mathfrak{M}_{i+1} = HOD^{\mathfrak{M}_i}$ $i \leq n-1$.

G. TAKEUTI, Gödel Numbers of Product Spaces.

Let $\varkappa, \lambda$ be infinite cardinals and $\lambda \leq \varkappa$. We define two Gödel numbers of ${}^{\varkappa}\lambda$ by

$$g(\varkappa,\lambda) = \min \{|M^*| \mid M \text{ is major in } {}^{\varkappa}\lambda\}$$
$$d(\varkappa,\lambda) = \min \{|M| \mid M \text{ is major in } {}^{\varkappa}\lambda\},$$

where $M^* = \{f|\alpha \mid f \in M \text{ and } \alpha < \varkappa\}$. Gödel's Ax. is equivalent to $g(\varkappa,\varkappa) = \varkappa$ and $d(\varkappa,\varkappa) = \varkappa^+$ for every regular cardinal $\varkappa$. I discussed the upper bounds and the lower bounds of $g(\varkappa,\lambda)$ and $d(\varkappa,\lambda)$ and pointed out the strange situation of the lower bound of $g(\varkappa,\omega)$ and presentd the open problem: Is $g(\varkappa,\omega) = 2^{\varkappa}$? Then we fixed $\varkappa$ and discussed the bevavior of $g(\varkappa,\lambda)$ and $d(\varkappa,\lambda)$ when λ goes from ω to $\varkappa$. We also presented an open problem "Is there any model of ZFC and $g(\varkappa,\lambda^+) > g(\varkappa,\lambda)$ or $d(\varkappa,\lambda^+) > d(\varkappa,\lambda)$?" We dicussed that Gödel's Ax. and Martin type Ax. are opposite to each other. At the end, I generalized my previous axiom on power set. The new axiom implies that $2^{\aleph_\alpha}$ is always a limit cardinal.

J. TRUSS, An Increasing Sequence of Degrees of Constructibility.

Balcar and Hajek have given a construction of a model of ZFC in which there is a sequence $(c_n | n \in \omega)$ of degrees of constructibility of reals, and degrees d,e such that

$$\text{(i)} \ (\forall i,j)\ (i < j \rightarrow c_i < c_j < d,e)$$
$$\text{(ii)} \ (\forall c)\ (c \leq d,e \rightarrow (\exists n)\ c \leq c_n)$$

where the quantifiers ranges over degrees of sets of ordinals. Their construction required the assumption that $\omega_1^L < \omega_1$. We show that the situation described above occurs in L[t] where t is a Cohen L-generic real. In addition, Balcar and Hajek have remarked the following. In L[t], there is a set r ($r = \{\xi \subseteq \omega \mid (\exists n) \deg \xi \leq c_n\}$) such that $(\forall i)\ c_i \leq \deg r \leq d,e$ and such that $(\forall \varkappa)\ ((\forall i)\ c_i \leq \deg \varkappa \leq d,e \rightarrow \deg r \leq \deg \varkappa)$. However, by careful choice of c_i,d,e and s, it may be arranged that $s \subseteq \omega$ and $(\forall i)\ c_i < \deg s$ and r,s have incomparable degrees.

E. WALZERMELODIE, A Dozen New Types of Large Cardinals.

ERASABLE: They can be quietly rubbed out after a contradiction.
INOPERABLE: Recently withdrawn from the market.
INDECIPHERABLE: They never Xerox well.
INELUCIDABLE: Only for the Cabal.
IMMEASURABLE: They won't sit still.
IMPREGNABLE: They never seem to have any consequences.
INSATIABLE: There never are enough models.
INCONTINENT: Whenever you try to prove something, they let go.
INEDIBLE: No one will swallow them.
INTOLERABLE: For everyone of yours there is a larger one of theirs.
INDEFATIGABLE: They show up at every meeting.
UNREMARKABLE: You didn't think of them.
INEXORABLE: Watch out! Here they come!

Karsten Steffens (Hannover)