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MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

T a g u n g s b e r i c h t 21/1977

Funktionalgleichungen

22.5. bis 28.5.1977

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The 15th International Symposium on Functional Equations was held from May 22 to May 28, 1977 in Oberwolfach, Germany. The organizational committee consisted of J.Aczél (Waterloo), W.Benz (Hamburg), and A.M.Ostrowski (Basel). The meeting was opened by J.Aczél who also took the occasion to transmit to Prof.G.Aumann the best wishes of the participants of the meeting for his 70th birthday.

Fortunately this time several participants could come from Hungary, Czechoslovakia, Poland, and Austria, while to our great regret none of the several experts in this field could come from Roumania and the U.S.S.R. The Symposium was attended by 48 participants from 12 countries.

Among the fields strongly represented were iterative equations, equations of classical and generalized types, and functional equations for multiplace functions. Cross relations with modern geometry, general and geometric algebra, topology, and functional analysis were extensively discussed. Particular accent was put on applications to economics, information theory, and statistics.

Solutions of different problems, formulated at this and previous meetings, were presented in the talks and discussion sessions.

In his closing remarks Prof. Ostrowski has thanked for the warm hospitality displayed by the Institute. The participants expressed their wish that the next functional equations meeting in Oberwolfach should take place June 3 - 9, 1979.

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List of participants

J.Aczél, Waterloo, Ont., Canada
N.L.Aggarwal, Besançon, France
R.Artzy, Haifa, Israel
G.Aumann, München
K.Baron, Katowice, Poland
W.Benz, Hamburg
S.Bilinski, Zagreb, Yougoslavia
B.Choczewski, Krakow, Poland
Z.Daróczy, Debrecen, Hungary
T.M.K.Davison, Hamilton, Ont.,Canada
W.Eichhorn, Karlsruhe
K.Endl, Giessen
W.Gehrig, Karlsruhe
R.Ger, Katowice, Poland
R.Graw, Marburg
H.Haruki, Waterloo, Ont., Canada
W.Herget, Braunschweig
H.-H.Kairies, Clausthal-Zellerfeld
M.Kuczma, Katowice, Poland
L.Losonczi, Debrecen, Hungary
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W.Maier, Mainz
J.Matkowski, Bielsko-Biala, Poland
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F.Neuman, Brno, Czechoslovakia
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C.F.Picard, Paris
J.Rätz, Bern
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L.Reich, Graz, Austria
D.C.Russell, Toronto, Canada
J.V.Ryff, Storrs, Conn., USA
E.Schröder, Hamburg
P.Schroth, Braunschweig
J.Schwaiger, Graz, Austria
H.Schwerdtfeger, Montreal, Canada
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K.Strambach, Erlangen
J.Tabor, Krakow, Poland
J.Targonski, München
G.Targonski, Marburg
T. van der Pyl, Paris
W.Walter, Karlsruhe
G.Zimmermann, Bochum

Summaries of talks

A.OSTROWSKI (Basel): A functional inequality in the theory of Diophantine approximations.

The talk dealt with the functional inequality:

$$A(x) \leq \alpha A(y) + \beta x \tau(x), \quad \alpha > 1, \quad \beta > 0, \quad A(x) \leq x, \quad \tau(x) \downarrow 0, \quad x \wedge y \geq 1 .$$

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Solutions were discussed in four cases $(x \uparrow \infty)$

A) $\tau(x) = O(\frac{1}{x})$, $\alpha = 1$.

B) $\tau(x) = O(\frac{1}{x})$, $\alpha > 1$.

C) $\tau(x) \leq \frac{g}{2\alpha}\tau(gx)$, $g > 2\alpha$ $(x \geq x_0 > 1)$.

D) $\tau(x) = O(\frac{1}{x^\delta})$, $\delta > 1$.

L.REICH (Graz): Analytische und fraktionelle Iteration und Differenzenrechnung.

Es sei $F: x \mapsto F(x) = Ax + p(x)$ ein Automorphismus des Ringes $C[x_1,\dots,x_n] = C[x]$ der formalen Potenzreihen in $x = {}^T(x_1,\dots,x_n)$; $\rho_1,\dots,\rho_n$ seien die Eigenwerte von A, $\Lambda = (\ln \rho_1,\dots,\ln \rho_n)$ sei eine feste Wahl der Logarithmen der ρ_j. In der Theorie der analytischen und fraktionellen Iterationen der Automorphismen hat sich folgendes Existenzkriterium als grundlegend erwiesen (REICH-J.SCHWAIGER): F besitzt eine zu Λ gehörige analytische Iteration genau dann, falls F zu einer sog. bezüglich Λ glatten Normalform konjugiert ist. Im Anschluss an eine Idee von D.C.LEWIS wird ein ziemlich einfacher Beweis skizziert, der davon ausgeht, daß die Funktionalgleichung der analytischen Iteration $F_t \circ F_{t'} = F_{t+t'}$, $F_1 \overset{!}{=} F$ von F als notwendige Bedingung $F_{t+1} = F \circ F_t$ enthält, was einem unendlichen rekursiven System von Differenzengleichungen äquivalent ist. Geht man von F in glatter Normalform bezüglich Λ aus und verwendet man bekannte Sätze über die Struktur glatter Normalformen, so sondert man eine analytische Lösung dieses Systems von Differenzengleichungen aus, von der es sich mittels vollständiger Induktion und eines einfachen Satzes über Polynome (in t,t') zeigen läßt, daß sie tatsächlich eine Iteration von F zu Λ liefert.

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J.SCHWAIGER (Graz): Über die Funktionalgleichung $N \circ T = T \circ M$ für formale Potenzreihen.

Ausgehend von [1] werden Zusatzmonome eines festen Grades m auch im nicht-kontrahierenden Fall als Basisvektoren eines Unterraumes Q_m von P_m, dem Raum der homogenen Polynome vom Grad m mit Koeffizienten aus K^n, gedeutet. Dabei ist $Q_m = \bigcup_{\ell \geq 0} \operatorname{Ker} \Psi^{\ell}_{J,m}$ und $\Psi_{J,m}$ ein K-linearer Endomorphismus von P_m, der in kanonischer Weise mit der Jordan'schen Normalform J des Linearteils A der formalen Reihe $F = AX + \mathcal{P}(X) \in (K[X])^n$ verbunden ist.

Unter Verwendung dieser Begriffe ergibt sich wiederum

SATZ 1: Jedes $F = AX + \mathcal{P}(X)$ ist konjugiert zu einer Normalform $N = JX + \mathcal{N}(X)$, so daß mit $\mathcal{N}(X) = \sum_{i \geq 2} n_i(X)$ und homogenen Polynomen n_i vom Grad i alle n_i sogar in Q_i liegen.

Daraus ergibt sich folgendes, in der Theorie der analytischen Iterationen verwendete, Ergebnis:

SATZ 2: Sind N und M Normalformen im oben beschriebenen Sinn und vertauscht der Automorphismus T, N und M, so hat T die folgende Gestalt: $T = BX + \sum_{i \geq 2} t_i(X)$.

Dabei gilt noch: (i) $B \cdot J = J \cdot B$ und (ii) $t_i \in Q_i$ für $i \geq 2$.

Literatur: [1] L.REICH: Das Typenproblem bei formal-biholomorphen Abbildungen mit anziehendem Fixpunkt. Math.Ann. 179 (1969), 227-250.

K.BARON (Katowice): On a system of functional equations with measures as unknown functions.

Given a non-empty set X and a σ-algebra $\mathcal{A}$ of its subsets the problem of the existence, uniqueness and properties of solutions of the system

$$\mu_i(A) = \sum_{j=1}^{M} \sum_{k=1}^{N} \int_{T_k(A)} f_{i,j,k} \, d\mu_j \circ S_k + \nu_i(A), \quad i \in \{1,\dots,M\},$$

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is considered, where ν_i are given measures on $\mathcal{A}$, $f_{i,j,k}: X \to \mathbb{C}$ are given measurable functions whereas S_k and T_k are given self-mappings of $\mathcal{A}$, $i,j \in \{1,\dots,M\}$, $k \in \{1,\dots,N\}$.

R.GRAW (Marburg): Strukturelle Ähnlichkeiten von endlichen und kompakten Orbits.

Es wird die Frage untersucht, unter welchen Bedingungen in Orbits Periodizitäten auftreten. Der Orbitbegriff, der die iterativen Eigenschaften einer Funktion beschreibt, wurde wahrscheinlich von Kuratowski (1924) und Whyburn (1942) eingeführt. Die Struktur eines kompakten Orbits wird charakterisiert durch folgenden

SATZ I: Sei $f: X \to X$ stetig und X ein kompakter Orbit (d.h.: $x,y \in X\ \exists n,m \in N: f^n(x) = f^m(y)$) und A1-Raum. Dann gibt es $Z \subseteq X$, $Z \neq \emptyset$, mit:

a) Für $x \in X$, $z \in Z$ folgt $\overline{S}_x \supseteq Z$, $\overline{S}_z = Z$.

b) Z ist die Menge aller fastperiodischen Elemente von X, d.h.: $\forall x \in X\ \forall U \subseteq X$, U Umgebung von x, $\exists\, n \in \mathbb{N}$: $f^n(x) \in U$.

c) Z ist kompakt und invariant (d.h.: $f(Z) = Z$) und $x \in X\ \exists\, n \in \mathbb{N}$: $f^n(x) \in Z$.

Unter welchen weiteren Bedingungen ist die Menge Z ein Zyklus (d.h.: $\forall\, z \in Z\ \exists n \in \mathbb{N}: f^n(z) = z$), wie dies bei endlichen Orbits immer der Fall ist? Unter den Voraussetzungen von Satz I werden drei weitere Sätze angeführt (davon bauen zwei bzgl. ihres Beweises auf Satz I auf): Falls f injektiv ist, so ist X ein Zyklus. Falls f offen ist oder gleichgradige Iterierte besitzt (X metrischer Raum), so hat X einen Zyklus.

W.MAIER (Mainz): Mertens'sche Summen.

Aus den Möbiusschen Zahlen μ_λ bildete F.Mertens für $0 < x$

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$\sum_{\lambda\le x}\mu_\lambda = M(x)$. Durch Benutzung von Farey'schen Brüchen sollen Funktionalgleichungen für $M(x)$ hergeleitet werden. Die Brüche $\{\rho_\nu\}$; $0 < \rho_\nu \le 1$, $\rho_\nu = \frac{b_\nu}{c_\nu}$; $(b_\nu, c_\nu) = 1$ heißen für $c_\nu \le \ell$ Farey-Brüche ℓ-ter Ordnung, und deren Anzahl ist $n = \sum_{\lambda=1}^{\ell}\varphi_\lambda$; $\varphi_1 = 1$, $\sum_{d/\lambda}\varphi_d = \lambda$. Für analytische $f(t) \not\equiv \infty$ gilt $\sum_{\nu=1}^{\ell} f(t+\rho_\nu) = \sum_{k=1}^{\ell} M(\frac{\ell}{k}) \sum_{\kappa=1}^{k} f(t+\frac{\kappa}{k})$. Um die Potenzsummen $\sum_{\nu=1}^{n}\rho_\nu^h = s_h$; $h = 1,2,\dots$ aus Bernoullischen Polynomen darzustellen, wird

$$s_h = 1 + \sum_{k=1}^{\ell} k^{-h}\frac{B_{h+1}(k) - B_{h+1}}{h+1}.$$

Als symmetrische Polynome in $\{\rho_\nu\}$ sind diese s_h für $h \le n+1$ algebraisch abhängig. Nach Ausschaltung der $\{s_h\}$ bleiben die $\{M(\frac{\ell}{k})\}$ verknüpft durch das Verschwinden eines Polynoms vom Grade $[\frac{\ell+1}{2}]$, dessen Beiwerte aus den $\{B_\nu\}$; $\nu \le n$ gebildet sind. - Als lineare Funktionalgleichung für $M(x)$ besteht andererseits mit $\hbar(\sigma,t) = \sum_{g=0}^{\infty}(t+g)^{-\sigma}$;

$$1<\mathrm{Re}\,\sigma<\mathrm{Re}\,t: \sum_{k=1}^{\ell} M(\tfrac{\ell}{k})\{k^\sigma \hbar(\sigma,tk) - \sum_{g=\nu}^{\alpha} k^{-g}\binom{-\sigma}{g}\hbar(\sigma+g,t)\frac{B_{g+1}(k+1)-B_{g+1}}{g+1}\} = t^{-\sigma}.$$

Schrifttum: M.Mikolás: Farey series.. (I), Act.Sci.Math, (Szeged) 13, 93-117.

B.CHOCZEWSKI (Krakow): <u>One parameter families of solutions of a linear functional equation.</u>

The linear homogeneous functional equation for the function φ:

(1) $\qquad \varphi(f(x)) = g(x)\varphi(x)$

is considered in the case, where its continuous solution in an interval $[0,b)$ $(f \in C[0,b),\ f(0) = 0,\ 0 < f(x) < x$ in $(0,b))$ depends on an arbitrary function. It is proved by D.Brydak and the author (in a paper prepared to print) that, if $g(x) = q(x)r(x)$ in $[0,a)$ with suitably chosen functions q and r, then there is a unique one-parameter family of continuous solutions of (1) in $[0,b)$, which are asymptotically comparable at the origin with a solution v of the functional inequality

$$v(f(x)) \le q(x)v(x).$$

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J.V.RYFF (Storrs): Über die Funktionalgleichung $af(ax) + bf(bx+a) = bf(bx) + af(ax+b)$.

The functional equation described above arises in the study of an extreme point problem for positive operators on the space $L^1(0,1)$. Background information may be obtained by consulting the Bull.Amer.Math.Soc. 82 (1976), 325-327 paper of the speaker.
The equation is viewed from several perspectives. First is the extension problem: If a solution exists on an interval, can the function be extended while remaining a solution ? Second, one wishes to consider examples of solutions in an attempt to decide whether or not enough concrete solutions exist to span the space of all solutions. The final objective is to select the extreme points from a certain convex set of solutions.

W.HERGET (Braunschweig): Über die Funktionalgleichung $f(x) = d^{m-1} \sum_{i=0}^{d-1} f(\frac{x+i}{d})$.

Die BERNOULLI-Polynome $B_m: \mathbb{R} \to \mathbb{R}$ sind die bis auf eine additive Konstante eindeutigen Polynomlösungen der Differenzengleichungen

$$\text{(D)} \qquad f(x+1) - f(x) = mx^{m-1} .$$

Sie erfüllen ein Multiplikationstheorem der Form

$$\text{(M)} \qquad f(x) = d^{m-1} \sum_{i=0}^{d-1} f(\frac{x+i}{d})$$

für jedes $x \in \mathbb{R}$ und $d \in \mathbb{N}$.
Kennzeichnungen der $B_m: \mathbb{R} \to \mathbb{R}$ durch (M) sind von ANASTASSIADIS und KAIRIES angegeben worden. Dabei zeigt sich, daß (M) eine größere kennzeichnende Kraft besitzt als (D). DICKEY/KAIRIES/SHANK haben dann diese Funktionalgleichungen in endlichen Ringen betrachtet, und zwar in einigen Spezialfällen (m=0, d=2; m=1) .
In den verbleibenden Fällen ist die Behandlung von (D) einfach,

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die Kennzeichnung durch (M) jedoch erheblich schwieriger. Tatsächlich stellt sich schließlich heraus, daß das Multiplikationstheorem (M) hier "schwächer" ist als die Differenzengleichung (D).

P.SCHROTH (Braunschweig): Functional equations in Nörlund's theory of difference equations.

N.E.Nörlund defined the principal solution $\phi: \mathbb{R} \to \mathbb{R}$ of the difference equation $\forall (x,y) \in \mathbb{R}\times\mathbb{R}_+ : \frac{1}{y}[g(x+y,y) - g(x,y)]=\phi(x)$ (D)
by $\forall (x,y) \in [b,\infty)\times\mathbb{R}_+$:

$$f_N(x,y) := \lim_{s\to 0+} \left(\int_a^\infty \phi(t)e^{-st}\,dt - y\sum_{n=0}^{\infty}\phi(x+ny)e^{-s(x+ny)}\right)$$

with suitable $a,b \in \mathbb{R}$.
Every principal solution f_N fulfills the multiplication formula

$$\forall p \in \mathbb{N}_+ \ \forall (x,y) \in \mathbb{R}\times\mathbb{R}_+ : f_N(x,y) = \frac{1}{p}\sum_{k=0}^{p-1} f_N(x+ky,py) \qquad \text{(M)}.$$

Special forms of (M) are well known, e.g. in the theory of the Γ- and the Ψ-function and in connection with the Bernoulli-polynomials. We study (M) in the following directions:

a) Characterization of functions by (M).
b) Equivalent characterization of (M) by means of Fourier expansion.
c) Connection between (M) and (D).
d) New definition of a principal solution, which includes Nörlund's, with the aid of (M).

H.HARUKI (Waterloo): A characteristic property of orthogonal pencils of circles from the standpoint of conformal mapping.

The following theorem holds: For orthogonal parabolic or elliptic and hyperbolic pencils of circles, the four vertices of a curvilinear rectangle formed by any four members arbitrarily chosen are concyclic. The purpose of this talk is to prove that

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the above property is a characteristic property of orthogonal pencils of circles from the standpoint of conformal mapping.

References:

[1] Aczél,J. and McKiernan, M.A., Math.Nachr. 33, 315-337 (1967).
[2] Haruki,H., Duke Math.J.36, 257-299 (1969).
[3] Haruki,H., Ann.Polon.Math.31, 171-177 (1975).
[4] Schwerdtfeger,H., Geometry of complex numbers, University of Toronto Press, 1962.

L.LOSONCZI (Debrecen): On homogeneous mean values.

Let r be a fixed nonnegative integer and I an interval. Further let $f: I^{r+1} \to (0,\infty)$ a continuous function and $\varphi: I \to (-\infty,\infty)$ differentiable on I with non-vanishing derivative. If $x = (x_1,\dots,x_{n+r}) \in I^{n+r}$ the quantity

$$(1) \qquad \mathfrak{M}_\varphi(x)_f = \varphi^{-1}\left(\frac{\sum_{i=1}^{n} f(x_i,\dots,x_{i+r}).\varphi(x_{i+r})}{\sum_{i=1}^{n} f(x_i,\dots,x_{i+r})}\right)$$

is a mean value of $x_{r+1},\dots,x_{r+n}$ since $\min_{1\le i\le n} x_{i+r} \le \mathfrak{M}_\varphi(x)_f \le \max_{1\le i\le n} x_{i+r}$. The problem of the equality of two such means is solved and the mean values (1) which are homogeneous functions of degree one on $I = (0,\infty)$ are determined. We also investigate the Minkowski inequality for a homogeneous mean and the question of the comparison of two homogeneous means.

J.TABOR (Krakow): Monotonic solutions of the translation equation.

Let X be a linearly ordered set and G a linearly ordered group. We shall say that a function $F: X \times G \to X$ satisfying the translation equation

$$(1) \qquad F(F(x,\alpha),\beta) = F(x,\beta\alpha), \quad x \in X, \ \alpha,\beta \in G,$$

is a monotonic solution of (1) whenever the functions

$F(.,\alpha)$ and $F(x,.)$ are monotonic (for all $\alpha \in G$ and $x \in X$, respectively). If F is a monotonic solution of (1), then $F(.,\alpha)$ must be increasing but $F(x,.)$ may be either increasing or decreasing.
As it is known, a transitive solution of (1) can be written in the form

(2) $$F(x,\alpha) = g^{-1}(\alpha g(x)) \qquad x \in X,\ \alpha \in G,$$

where g is a bijection of the left cosets of G with respect to some subgroup G^* onto X.
It can be proved that a transitive solution F of (1) is monotonic iff it can be written in the form (2), where the subgroup G^* is convex and the function g is monotonic.

J.RÄTZ (Bern): When is a bijective homomorphism an isomorphism?

1. The answer is affirmative, e.g., for i) groupoid homomorphisms, ii) isotone mappings between totally ordered sets, and iii) continuous mappings from compact onto Hausdorff topological spaces or, equivalently, for iii') open mappings from Hausdorff onto compact topological spaces. We look for general sufficient conditions for the answer to be affirmative.
If A is a nonempty set and I a set, then a subset ρ of A^I is called a relation on A of type I and the pair (A,ρ) a relational system of type I. If $f: A \to B$, (A,ρ) and (B,σ) are relational systems of type I, and the mapping $f^I: A^I \to B^I$ is defined by $f^I(a_i)_{i\in I} := (fa_i)_{i\in I}$, then f is said to be a

a) homomorphism iff $f^I[\rho] \subset \sigma$,

b) full homomorphism iff $f^I[\rho] = \sigma \cap f^I[A^I]$,

c) isomorphism iff f is bijective and f and f^{-1} are homomorphisms.

It turns out that a bijective homomorphism f is an isomorphism iff it is full, i.e. iff $f^I[\rho] = \sigma$ holds.

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<u>Theorem 1.</u> Let $\Omega_I(A) \subset \mathcal{P}(A^I)$, $\Omega_I'(B) \subset \mathcal{P}(B^I)$ such that, for $g: B \rightarrowtail\!\!\twoheadrightarrow A$, $\sigma \in \Omega_I'(B)$ implies $g^I[\sigma] \in \Omega_I(A)$. If $\sigma \in \Omega_I'(B)$ and $\rho \in \Omega_I(A)$ maximal with respect to set inclusion, then any bijective homomorphism from (A,ρ) onto (B,σ) is an isomorphism.

<u>Theorem 2.</u> If $f: (A,\rho) \rightarrowtail\!\!\twoheadrightarrow (B,\sigma)$ and $g: (B,\sigma) \rightarrowtail\!\!\twoheadrightarrow (A,\rho)$ are bijective homomorphisms and ρ is finite, then f is an isomorphism.

2. <u>Applications to topological spaces.</u> For $X,Y \neq \emptyset$ put $A := \mathcal{P}(X)$, $B := \mathcal{P}(Y)$. Topologies are relations on A,B of type $\{1\}$. $\tilde{f}: X \rightarrowtail\!\!\twoheadrightarrow Y$ uniquely induces $f: A \rightarrowtail\!\!\twoheadrightarrow B$ by $f(X') := \tilde{f}[X']$ for all $X' \subset X$. f is a homomorphism iff $\tilde{f}$ is an open mapping. Now our initial statement iii') follows from Theorem 1; "Hausdorff" even could be replaced by "KC" (every compact set is closed). A famous example by C.Kuratowski (Fund.Math. 2 (1921), 158-160) shows that finiteness of ρ is essential in Theorem 2.

3. <u>Applications to partially ordered sets.</u> Here we put $I = \{1,2\}$. Partial orderings are relations of type $\{1,2\}$, total orderings are maximal partial orderings, and the homomorphisms are precisely the isotone mappings. Now Theorem 1 yields statement ii) at the beginning while Theorem 2 includes results obtained by A.Abian (Amer.Math.Monthly 77 (1970), 1092-1094).

4. <u>Applications to partial groupoids.</u> It is very natural to interprete a binary partial operation $\cdot$ on A as a relation of type $\{1,2,3\}$. The homomorphisms of partial groupoids $f: (A,\cdot) \to (B,\cdot)$ are characterized by $a_1,a_2,a_1a_2 \in A \Rightarrow fa_1 \circ fa_2 \in B$, $f(a_1a_2) = fa_1 \circ fa_2$. Then each of the following conditions is sufficient for a bijective homomorphism $f: A \longrightarrow B$ to be an isomorphism:

I) $b_1,b_2,b_1b_2 \in B \Rightarrow f^{-1}b_1 \circ f^{-1}b_2 \in A$.

II) There exists a bijective homomorphism $g: B \rightarrowtail\!\!\twoheadrightarrow A$, and the partial operation $\cdot$ on A has finite domain.

III) $(A,\cdot)$ is a groupoid (cf. statement i) above).

IV) f is the restriction of an injective groupoid homomorphism $f: G \to H$.

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V) (A,·) and (B,·) are weak subalgebras of commutative semigroups, and there exists a positive integer n such that

V_1) for every $a \in A$ there is a unique $a' \in A$ with $a'^n = a$,

V_2) for every $b \in B$ there is a unique $b' \in B$ with $b'^n = b$,

V_3) $a_1, a_2 \in A \Rightarrow a_1' a_2' \in A$.

Remark. The restriction of the group epimorphism $h(x) := e^{ix}$ ($x \in \mathbb{R}$) to the interval $A := [0,\pi]$ shows that "injective" can not be omitted in IV) and that the uniqueness requirement in V_2) is essential.

N.L.AGGARWAL (Besançon), C.F.PICARD (Paris): Functional equations and information measures with preference.

In Information Theory and Questionnaire Theory we come across three measures of information: weighted entropy (entropy of Shannon with utility) of Belis and Guiaşu, inaccuracy of Kerridge and preference and normalized preference of questionnaires. These measures, though different, have analogous properties. In this paper we unify these measures and define preference as a positive measure (additive) V. This generalization yields the following considerations:

- To study questionnaires in which the valuations are the positive measures, not necessarily probabilities.
- To replace the usual additivity of two experiments $\mathcal{L}$ and $\mathcal{B}$ by a weighted mean:

(1) $$H(\mathcal{L}\mathcal{B}) = V(\mathcal{B})H(\mathcal{L}) + V(\mathcal{L})H(\mathcal{B}) .$$

Shannon's case corresponding to $V(\mathcal{L}) = V(\mathcal{B}) = 1$.

- To form a functional equation from (1) while using the sum property.
- To solve the functional equation

$$f(pq,uv) = vf(p,u) + uf(q,v), \quad 0<p,q<1, \quad u,v \geq 0$$

under conditions quite different from those of Kannappan.

- An analogous treatment is given for the entropy of β-type with preference, constructed from the solution of the functional equation

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$$f(pq,uv) = vf(p,u) + up^{\beta-1}f(q,v), \quad \beta \in \mathbb{R}, \ \beta \neq 1.$$

- Applications are given for the case when the preference becomes a probability.

S.BILINSKI (Zagreb): Ein System von Funktionalgleichungen und seine Anwendungen auf einige geometrische Probleme.

Es wird das Problem gestellt, alle Funktionen F(x,y) zu finden, welche in bezug auf eine endliche diskontinuierliche Transformationsgruppe der Zahlenebene invariant sind, wobei jede Transformation der Gruppe einen bestimmten Bereich dieser Ebene auf sich abbildet. Das Problem wird gelöst durch die allgemeine Lösung eines Systems von Funktionalgleichungen. Es sei nun eine Ebene durch eine kontinuierliche ν-fach transitive Transformationsgruppe bestimmt, und ein primitives Polygon dieser Ebene als eine diskrete, endliche, zyklisch geordnete Punktmenge von $\nu+1$ Punkten erklärt, wobei primitive Polygone zweiter Art durch zwei unabhängige invariante Parameter bestimmt sind. Die Lösung des erwähnten Systems von Funktionalgleichungen ergibt die Möglichkeit, die allgemeine Form von skalaren Invarianten für primitive Polygone zweiter Art in jeder Ebene zu bestimmen.

H.SCHWERDTFEGER (Montreal): Eine Verallgemeinerung des Doppelverhältnisses.

Bekanntlich ist das Doppelverhältnis die im wesentlichen einzige projektive Invariante von vier Punkten auf der Geraden. In Verallgemeinerung des Doppelverhältnisses hat A.F.Möbius (J. Reine Angew. Math.4 (1828) 101-139, oder Ges. Werke, Bd.I, 445-480) projektive Invarianten von fünf Punkten allgemeiner Lage in der Ebene P_2 konstruiert. Mit Hilfe eines allgemeinen Satzes (Aequationes Math. 14 (1976) 105-110) wird die im wesentlichen einzige 5-Punkt Invariante $P_2^5 \to P_2$ als ein

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variabler Punkt in P_2 hergeleitet und damit bewiesen, daß zwei der (skalaren) Möbius-Invarianten das vollständige System aller projektiven 5-Punkt Invarianten in der Ebene erzeugen.

K.STRAMBACH (Erlangen): Rechtsdistributive Quasigruppen.

Es wurden alle topologischen rechtsdistributiven Quasigruppen angegeben, deren von den Rechtstranslationen erzeugte Automorphismengruppe als Zusammenhangskomponente eine quasieinfache Liegruppe besitzt.

W.BENZ (Hamburg): Eine Funktionalgleichung im Zusammenhang des hyperbolischen Flächeninhaltes.

Gegeben sei das Innere E des Einheitskreises

$$E = \{z \in \mathbb{C} \mid |z| < 1\},$$

und weiterhin die Gruppe B aller orientierungstreuen hyperbolischen Bewegungen von E. Ist dann $\psi: E \to \mathbb{R}$ eine Funktion, die der Funktionalgleichung

$$\psi(f(z)) \cdot |f'(z)|^2 = \psi(z)$$

für alle $z \in E$, $f \in B$ genügt, so gilt $\psi(z) = \frac{1}{(1-|z|^2)^2}$.
Mit der zusätzlichen Voraussetzung der Existenz von ψ_x, ψ_y in E wurde dieses Resultat von H.Schwerdtfeger in seinem Buch "Geometry of complex numbers" (Toronto 1962) hergeleitet, wobei es dazu diente, ein additives und B-invariantes Maß für den hyperbolischen Flächeninhalt angeben zu können,

$$\mu_o(M) = \iint_M \frac{4\,dx\,dy}{(1-(x^2+y^2))^2} .$$

Wir zeigen darüber hinaus in einem elementaren Beweis, daß alle solchen Maße μ bis auf ein festes $k > 0$ durch $\mu = k\mu_o$ gegeben sind.

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R.ARTZY (Haifa): Inverse-cycles.

In a quasigroup $(Q,\cdot)$, let e be a fixed element, and $J: Q \to Q$, $x \to xJ$, $x\cdot xJ = e$. If, for all $x \in Q\setminus\{e\}$, $xJ^n = x$ $(n \in Z^+)$ and $xJ^m \neq x$ for $m < n$ $(m \in Z^+)$, we say that Q consists of e and e-cycles of length n. In particular, if $(Q,\cdot)$ is a loop with unit element e, the "e-cycles" become "inverse-cycles". For finite loops, the following holds: A loop obeying the law $(xy)z = e \iff x(yz) = e$ ("weak inverse loop") can consist of e and m inverse-cycles of length n only if n divides $2m$.

E.LUKACS (Washington): Über eine Anwendung der Cauchyschen Funktionalgleichung in der Wahrscheinlichkeitstheorie.

Es seien $X_1,X_2,\ldots,X_n$ Zufallsvariable, die die gleiche symmetrische stabile Verteilungsfunktion $S_{\alpha\lambda}(x)$ haben. Ferner sei $\underline{Z} = (Z_1,Z_2,\ldots,Z_n)$ ein zufälliger Vektor, dessen Komponenten entweder alle nichtnegativ oder alle nichtpositiv sind, und es sei $F(z_1,z_2,\ldots,z_n)$ die Verteilungsfunktion des Vektors $\underline{Z}$. Es wird angenommen, daß $X_1,X_2,\ldots,X_n$ und $\underline{Z}$ unabhängig sind und man definiert

$$W = \sum_{j=1}^{n} \left(\frac{Z_j}{Z_1+Z_2+\ldots+Z_n}\right)^{1/\alpha} X_j\delta_j ,$$

wo die δ_j entweder $+1$ oder -1 sind. Dann hat auch W die Verteilungsfunktion $S_{\alpha\lambda}(x)$ für jedes $F(z_1,z_2,\ldots,z_n)$, das zu einem Vektor $\underline{Z}$ gehört, der den obigen Bedingungen genügt. Es gilt auch die Umkehrung dieses Satzes und es ergibt sich daher eine Charakterisierung der symmetrischen stabilen Verteilungen. Die Lösung dieses Charakterisierungsproblems führt letzten Endes auf die Cauchy'sche Funktionalgleichung.

F.STEHLING (Karlsruhe): The functional equation $f(yx) = g(y)f(x) + b(y)$ on a restricted domain.

The solutions of the functional equation

(1) $\qquad f(yx) = g(y)f(x) + b(y) \qquad (x > 0, y > 0)$

with $f: (0,+\infty) \to \mathbb{R}$, $g: (0,+\infty) \to \mathbb{R}$, $b: (0,+\infty) \to \mathbb{R}$, f strictly monotonic, are well-known (Hardy/Littlewood/Polya 1934). They are given by

(2) $\qquad f(t) = \alpha\log t + \beta,\ g(t) = 1,\ b(t) = \alpha\log t \qquad (t > 0),$

(3) $\qquad f(t) = \gamma t^\delta + \varepsilon,\ g(t) = t^\delta,\ b(t) = \varepsilon(1 - t^\delta) \qquad (t > 0),$

where $\alpha \neq 0$, β, $\gamma \neq 0$, $\delta \neq 0$, ε are arbitrary real constants.

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Some investigations in mathematical economics lead to the functional equation (1) where x is only allowed to vary in a certain interval $I \subseteq (0,+\infty)$. Nevertheless, it can be shown (using a result of Daróczy and Losonczi from 1967 about the solutions of the additive Cauchy equation on an arbitrary connected open set in $\mathbb{R}^2$):

Theorem: The solutions of

(4) $f(yx) = g(y)f(x) + b(y) \qquad (x \in I \subseteq (0,+\infty),\ y > 0),$

where $f: (0,+\infty) \to \mathbb{R}$, $g: (0,+\infty) \to \mathbb{R}$, $b: (0,+\infty) \to \mathbb{R}$, I is an interval and f is strictly monotonic, are given by (2) and (3).

R.W.SHEPHARD (Berkeley): Some functional equations for production theory.

A set valued mapping $P: x \in \mathbb{R}^n_+ \to P(x) \in 2^{\mathbb{R}^m_+}$, with axioms for this mapping, defines a production technology. Various assumptions on scaling of inputs lead to equations on sets $P(x)$ which serve to characterize scaling laws. From such equations, scaling properties are deduced.

W.EICHHORN (Karlsruhe): Functional equations for set-valued mappings.

The following theorems are of interest in the theory of production as developed by R.W.Shephard, R.Färe, and W.Eichhorn. Let the set-valued mapping or correspondence

$$P: \mathbb{R}^n_+ \to 2^{\mathbb{R}^m_+}, \quad x \mapsto P(x)$$

be given, where $P(x)$ is, e.g., the set of all output vectors $u \in \mathbb{R}^m_+$ which can be produced by the input vector x per unit time. Let L be the inverse

$$P^{-1}: \mathbb{R}^m_+ \to 2^{\mathbb{R}^n_+}$$

of P.

Theorem 1. The following assumptions on P and L:

(i) $P(\lambda x) = \psi(\lambda,x)P(x)$ for all $(\lambda,x) \in \mathbb{R}_{++} \times \mathbb{R}^n_+$, where $\psi: \mathbb{R}_{++} \times \mathbb{R}^n_+ \to \mathbb{R}_{++}$, $\psi(1,x) = 1$ for all $x \in \mathbb{R}^n_+$

(ii) $L(\mu u) = \chi(\mu,u)L(u)$ for all $(\mu,u) \in \mathbb{R}_{++} \times \mathbb{R}^m_+$, where $\chi: \mathbb{R}_{++} \times \mathbb{R}^m_+ \to \mathbb{R}_{++}$, $\chi(1,u) = 1$ for all $u \in \mathbb{R}^m_+$.

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(iii) $\lambda \mapsto \psi(\lambda,x)$ and $\mu \mapsto \chi(\mu,u)$ are strictly increasing from 0 to ∞

(iv) $P(x) \subseteq P(\lambda x)$ for all $\lambda \in [1,\infty[$.

(v) $x \in L(\nu u)$ yields $x \in L(u)$ for each $\nu \in [1,\infty[$

imply that both P and L are semi-homogeneous, i.e.,

$$P(\lambda x) = \lambda^{\varphi(x/|x|)} P(x) \qquad (\varphi(x/|x|) \in \mathbb{R}_{++}).$$

Theorem 2. The assumptions (i),(ii),(iii), and

(iv') $x \leq x^*$ yields $P(x) \subseteq P(x^*)$

imply that P is homogeneous of positive degree on $\mathbb{R}^n_{++}$.

W.GEHRIG (Karlsruhe): Mean values in economic distribution theory.

In several fields of economic distribution theory indices of societal income (ISI) play a central role. The value of such a function, which depends on the incomes of the participants of the economy, is interpreted as a representative income. Let x_i denote the income of person i in an economy. We assume $x_i \in [0,a]$ $(a > 0)$. Furthermore $\underline{x}$ denotes the vector $(x_1,\ldots,x_N)$.

Def.: An ISI is a function $f^N: [0,a]^N \to \mathbb{R}_+$ satisfying the axioms (1) - (6)

(1) $f^N \in C([0,a])$

(2) $f^N(b,\ldots,b) = b$ for all $b \in [0,a]$

(3) $\underline{x} \geq \underline{y} \Rightarrow f^N(\underline{x}) > f^N(\underline{y})$

(4) $f^N(x_1,\ldots,x_N) = f^N(x_{\pi(1)},\ldots,x_{\pi(N)})$ $\forall \pi$
(π a permutation of $\{1,\ldots,N\}$)

(5) $f^N(x_1,\ldots,x_k,x_{k+1},\ldots,x_N) = f^N(f^k,\ldots,f^k,x_{k+1},\ldots,x_N)$
for all $k \leq N$, where $f^k := f^k(x_1,\ldots,x_k)$

(6) $f^N(sx_1,\ldots,sx_N) = sf^N(x_1,\ldots,x_N)$ $\forall s \in (0,1]$, $x_i \in (0,a]$.

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Theorem: Axioms (1) - (6) are satisfied iff

$$f^N(x_1,\ldots,x_N) = \left[\frac{1}{N}(x_1^\alpha + \ldots + x_N^\alpha)\right]^{1/\alpha}$$

where α is a real positive constant.

(R.BÖRK-W.GEHRIG: Indices of sociatal income and income inequality. To appear in: Karlsruhe Symposium on Index Numbers, Springer-Verlag, 1977).

Z.MOSZNER (Krakow): Sur l'équation $f(x)f(y)f(-x-y)=g(x)g(y)g(-x-y)$.

On donne quelques théorèmes au sujet des fonctions f et g remplissants l'équation en titre. En particulaire on donne la solution générale de cette équation dans les classes des fonctions de R à C,

a) qui sont différentes de zéro en zéro $(f(0) \neq 0)$ et pour lesquelles les ensembles des racines sont symétriques par rapport à zéro,

b) qui appartiennent à la classe plus haut et sont continues,

c) qui sont continues et remplissent les conditions $f(0) = 1$ et $f(-x) = \overline{f(x)}$.

Z.DARÓCZY (Debrecen): Über die Funktionalgleichung der Additivität.

Es sei $\Gamma_n^o := \{(p_1,\ldots,p_n)\colon p_i > 0,\ \sum_{i=1}^n p_i = 1\}$, $(n = 2,3,\ldots)$.
Die Folge von Funktionen $I_n\colon \Gamma_n^o \to \mathbb{R}$ $(n = 2,3,\ldots)$ wird (k,ℓ)-additiv genannt, wenn für alle $(p_1,\ldots,p_k) \in \Gamma_k^o$ und $(q_1,\ldots,q_\ell) \in \Gamma_\ell^o$

$$I_{k\ell}(p_1q_1,\ldots,p_1q_\ell;\ldots;p_kq_1,\ldots,p_kq_\ell) = I_k(p_1,\ldots,p_k) + I_\ell(q_1,\ldots,q_\ell)$$

erfüllt ist. Die Folge $\{I_n\}$ hat die Summen-Eigenschaft, wenn es eine Funktion $f\colon]0,1[\to \mathbb{R}$ gibt, so daß

$$I_n(p_1,\ldots,p_n) = \sum_{i=1}^n f(p_i)$$

für alle $(p_1,\ldots,p_n) \in \Gamma_n^o$ und $n = 2,3,\ldots$ gilt. Es gilt der folgende

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Satz: Es sei $\{I_n\}$ eine Folge, die die Summen-Eigenschaft mit einem meßbaren f hat. Ist $\{I_n\}$ (2,2)-additiv, so gilt

$$I_n = -3AH_n^3 + \frac{9}{2}AH_n^2 - BH_n^1 + nC \qquad (n = 2,3,\dots)$$

wobei A,B,C beliebige Konstanten und $H_n^\alpha : \Gamma_n^o \to \mathbb{R}$ $(\alpha > 0)$ die Entropien vom Typ α sind.

Der Satz wurde mit Hilfe der Methode von A.Járai bewiesen.

H.ŚWIATAK: Simple regularity criteria for the solutions of functional equations. (Presented by B.Choczewski.)

The regularity criteria concern the solutions $f: \mathbb{R}^n \to \mathbb{R}$ of the functional equation of the type

$$(1) \qquad \sum_{i=1}^{m} \mu_i f(x + \varphi_i(t)) = f(x) ,$$

where $x \in \mathbb{R}^n$, $t \in \Delta \subset \mathbb{R}$ (Δ being an open interval), $\mu_i > 0$, $\sum_{i=1}^{m} \mu_i = 1$, $\varphi_i: \Delta \to \mathbb{R}^n$ and there exists an $\alpha \in \Delta$ such that $\varphi_i(\alpha) = 0$ for $i = 1,\dots,m$. They contain easily applicable conditions for every continuous solution of (1) to be a C^∞ function on $\mathbb{R}^n$ and for every locally integrable solution of (1) to be equal almost everywhere to a C^∞ solution.

J.ACZÉL (Waterloo): Funktionen des binomialen und des Potenzsummen Typs.

Satz 1. Es sei $(G,+)$ ein Gruppoid, $(R,+,\cdot)$ ein kommutativer Ring. Die allgemeinen Lösungen $f_k: G \to R$ des Funktionalgleichungssystems

$$f_k(x+y) = f_k(x) + f_k(y) + \sum_{j=1}^{k-1} \binom{k}{j} f_j(x) f_{k-j}(y) \qquad (x,y \in G)$$

der Funktionen des binomialen Typs sind durch

$$f_k(x) = k! \sum_{i_1+2i_2+\dots+ki_k=k} \prod_{j=1}^{k} \left(\frac{a_j(x)}{j!}\right)^{i_j} / i_j! \qquad \text{für alle } x \in G$$

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gegeben, wo a_j eine beliebige Lösung von

$$a_j(x+y) = a_j(x) + a_j(y) \qquad (x,y \in G)$$

ist $(j,k = 1,2,\ldots)$. (Vgl. J.Aczél - G.Vranceanu, Colloq.Math. 36 (1972), 371-383, für den Fall, wo G eine Halbgruppe und R ein Körper ist).

Insbesondere für $R = \mathbb{R}$, $G = \mathbb{R}$ oder $G = \mathbb{R}_+$, falls die f_k von einer Seite beschränkt sind auf einer Menge von positivem Maße, oder in jedem Falle für $G = \mathbb{N}$, werden alle Funktionen des binomialen Typs Polynome sein.

Satz 2. Es sei $(G,+)$ eine torsionsfreie divisible abelsche Gruppe (für alle $n \in \mathbb{N}$, $b \in G$ hat $na = b$ eine Lösung a).
Die allgemeinen Lösungen $g_k\colon \mathbb{N} \to G$ des Systems

$$g_k(x+y) = g_k(x) + g_k(y) + \sum_{j=1}^{k} \binom{k}{j} g_{k-j}(y)x^j \qquad (x,y \in \mathbb{N})$$

(erfüllt u.a. durch $g_k(x) = 1^k + 2^k + \ldots + x^k$) sind durch

$$g_k(x) = \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} x^{k+1-j} c_j$$

gegeben $(k = 0,1,2,\ldots)$, wo c_j $(j = 0,1,2,\ldots)$ beliebige Konstanten in G sind.

Ähnlicher Satz gilt auch, wenn der Wertebereich ein kommutativer divisibler Ring und der Bereich ein Untergruppoid der additiven Gruppe dieses Ringes ist.

Satz 3. Es sei $(R,+,\cdot)$ ein divisibler kommutativer Ring.
Die allgemeinen Lösungen $g_k\colon \mathbb{N} \to R$ von

$$g_k(x+y) = g_k(x) + g_k(y) + \sum_{j=1}^{k} \binom{k}{j} g_{k-j}(y)(xh)^j \qquad (x,y \in \mathbb{N})$$

$(h \in R$ eine Konstante) sind durch

$$g_k(x) = \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} x^{k+1-j} c_j h^{k-j} \qquad (x \in \mathbb{N})$$

gegeben, wo c_j beliebige Konstanten in R sind $(j,k = 0,1,2,\ldots)$.

Auch hier kann N durch ein Untergruppoid von $(R,+)$ ersetzt werden.

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L.REDLIN (Hamilton): Further results on Hosszú's functional equation.

The functional equation $f(x+y-xy) + f(xy) = f(x) + f(y)$ is studied for functions from a commutative ring R to an abelian group G ($f: R \to G$). Conditions are obtained under which solving the equation for functions on R can be reduced to solving the equation for functions on a certain homomorphic image of R.
This result is then applied to the case where R is a ring of cyclotomic integers.

M.KUCZMA (Katowice): Asymptotic properties of solutions of a functional equation.

Asymptotic behaviour at the origin is investigated for the continuous solutions $\varphi: X \to \mathbb{R}$ of the functional equation

$$\varphi[f(x)] = g(x)\,\varphi(x) .$$

Here $X = (0,A]$ or $(0,A)$ is a real interval, the functions $f,g: X \to \mathbb{R}$ are continuous, $0 < f(x) < x$ and $g(x) > 1$ in X.
Moreover, f and g are assumed to have the asymptotic properties

$$f(x) = x - ax^{m+1} + O(x^{m+1+\mu}),\quad g(x) = 1 + bx^{k} + O(x^{k+\kappa}),\quad x \to 0,$$

where a,b,k,m,κ,μ are positive constants, $k \le m$.
The corresponding inhomogeneous equation

$$\varphi[f(x)] = g(x)\,\varphi(x) + h(x),$$

where $h: X \to \mathbb{R}$ is continuous and

$$h(x) = cx^{q} + o(x^{q}),\quad x \to 0,\quad c,q \in \mathbb{R},$$

is also dealt with.

F.NEUMAN (Brno): Functional equations in the theory of differential equations.

A procedure that in the theory of linear differential equations of the n-th order $(n \ge 2)$ leads to functional equations was

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described. This procedure essentially occurs when we extend the local investigations of Kummer, Laguerre, Brioschi, Halphen, Forsyth, Stäckel, Lie and others to global transformations of the equations using a group-theoretic approach.

A.SKLAR (Chicago): <u>The embedding of functions in flows.</u>

A real flow on a (non-empty) set S is a family $\tilde{f} = \{f_\alpha \mid \alpha \text{ real}\}$ of functions mapping S into S such that f is closed under composition, and $f_\alpha \circ f_\beta = f_{\alpha+\beta}$ for all real α, β. A function g mapping S into S is embeddable in a real flow on S if there is such a flow $\{f_\alpha\}$ with $f_1 = g$. Using the theory of orbits (cf. R.Isaacs, Canadian J.Math. 2, 1950, 409-416; A.Sklar, AequationesMath. 12, 1975, 307-308) and the properties of ultrastable functions (cf. A.Sklar, Aequationes Math. 3, 1969, 118-129), we have obtained the following:

Theorem. Let g be a function mapping a non-empty set S into itself. Then g is embeddable in a real flow on S if and only if g satisfies the following conditions:

(1) g is ultrastable, i.e., the range of $g \circ g$ is the same as the range of g, and the restriction of g to its range is one-one;

(2) for each integer $m \geq 2$, the number of cyclic orbits of g of order m is either 0 or infinite;

(3) the number of acyclic orbits of g is either 0 or infinite.

Other types of flows are also considered, their relation to real flows investigated, and corresponding embeddability results obtained in certain cases.

B.SCHWEIZER: <u>When is $f(f(z)) = az^2 + bz + c$ for all z ?</u>
(Presented by A.Sklar.)

Recent work by R.Rice, A.Sklar, and the author on fractional iteration of functions has yielded, among others, the following

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result:

Theorem. Let Q be a quadratic polynomial defined on the entire complex plane. Then there exists no function f whatever such that $f(f(z)) = Q(z)$ for all complex z.

In proving the theorem, we write $Q(z) = az^2 + bz + c$, $a \neq 0$, b,c fixed, and define $\Delta(Q) = (b-1)^2 - 4ac$. If $\Delta(Q) \neq 4$, then Q has precisely one cyclic orbit of order 2. If $\Delta(Q) = 4$, then Q, though lacking cyclic orbits of order 2, has precisely 3 cyclic orbits of order 4 (the proof of this involves the demonstration that all the roots of a certain 12^{th} degree polynomial are distinct). The conclusion of the theorem then follows immediately from some standard results in orbit theory (cf. R.Isaacs, Canadian J.Math. 2, 1950, 118-129).

G.ZIMMERMANN (Bochum): Iterative roots of Čebyšev polynomials.

We consider the Čebyšev polynomials C_n, defined in the complex plane by: $C_0(z) = 2$, $C_1(z) = z$, $C_{n+2}(z) = zC_{n+1}(z) - C_n(z)$ for all integers $n \geq 0$. By an iterative root of order d of C_n, d a natural number, we mean a function f mapping the complex plane into itself whose d^{th} iterate is C_n. We have proved the following $(n \geq 2)$.

Theorem 1: Let p be a prime number, k a natural number. Then C_n has an iterative root of order p^k if and only if $n^p \equiv n \bmod p^{k+1}$ and $p^k \leq n-1$.

Theorem 2: Let $p_i^{\nu_i}$, $1 \leq i \leq k$, be distinct prime powers, $n^{p_i} \equiv n \bmod p_i^{\nu_i+1}$ for $1 \leq i \leq k$, $\prod_{i=1}^{k} p_i^{\nu_i} \leq n-1$. Then C_n has an iterative root of order $\prod_{i=1}^{k} p_i^{\nu_i}$.

Remark: If C_n has an iterative root of order d, then $d \leq \left[\frac{n}{2}\right] n$. These results remain valid if the polynomials C_n are restricted to the real line, or to the real interval $[-2,2]$.

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L.PAGANONI (Mailand): Cauchy's functional equation on semigroups.

Let G and H be semigroups and consider the Cauchy's equation on a restricted domain

$$(*) \qquad f(x\cdot y) = f(x)\cdot f(y)$$

where $(x,y) \in \Omega \subset G \times G$ and $f\colon G \to H$.
We look for some conditions under which a solution of (*) is a homomorphism of G into H.
The theorems we prove have some interesting corollaries which generalize some well-known results.

E.VINCZE: Über die Einführung der komplexen Zahlen mittels Funktionalgleichungen. (Presented by J.Rätz.)

Ausgangspunkt ist der bewertete Körper $\langle \mathbb{R},+,\cdot,\|\cdot\|\rangle$ der reellen Zahlen. Auf $\mathbb{R}^2$ sollen eine innere Verknüpfung $*$ und eine Bewertungsfunktion w so eingeführt werden, daß $\langle \mathbb{R}^2,+,*,w\rangle$ ein bewerteter kommutativer Körper wird, der $\mathbb{R}$ als Unterstruktur enthält. An verschiedenen Stellen in der Entwicklung werden Funktionalgleichungen des Cauchyschen Typs verwendet. Mit der Stetigkeitsforderung an $*$ werden die Möglichkeiten für $*$ gleich zu Beginn wesentlich eingeschränkt.

G.AUMANN (München): Map theoretical approach to topology.

The axioms of a (topological) neighborhood space $(X,\mathfrak{U})$, $\mathfrak{U}\colon X \to \mathfrak{P}(\mathfrak{P}(X))$, namely

(t_o): $\bigwedge_{x\in X} \mathfrak{U}(x) \neq \emptyset \wedge \bigwedge_{U,V\in\mathfrak{U}(x)} \bigvee_{W\in\mathfrak{U}(x)} W \subset U \cap V$;

(t_1): $\bigwedge_{x\in X} \bigwedge_{U\in\mathfrak{U}(x)} x \in U$;

(t_2): $\bigwedge_{x\in X} \bigwedge_{U\in\mathfrak{U}(x)} \bigvee_{V\in\mathfrak{U}(x)} \bigwedge_{y\in V} \bigvee_{W\in\mathfrak{U}(y)} W \subset U$,

allow a very suggestive map theoretical interpretation:
Using the relation

$$f \sim_{\mathfrak{U}(x)} g \;:\Longleftrightarrow\; \bigvee_{V\in\mathfrak{U}(x)} f|V = g|V$$

for the set F of all mappings $f,g,\ldots$ of X into Y with

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$X \neq \emptyset$ and $|Y| \geq 2$, one can show that

$(t_o) \Longleftrightarrow (t_o') :\Longleftrightarrow \bigwedge_{x \in X} \sim_{\mathfrak{U}(x)}$ is an equivalence relation in F;

$(t_1) \Longleftrightarrow (t_1') :\Longleftrightarrow \bigwedge_{f,g \in F} \bigwedge_{x \in X} f \sim_{\mathfrak{U}(x)} g \Rightarrow f(x) = g(x)$;

$(t_2) \Longleftrightarrow (t_2') :\Longleftrightarrow \bigwedge_{f,g \in F} \bigwedge_{x \in X} f \sim_{\mathfrak{U}(x)} g \Rightarrow \bigvee_{V \in \mathfrak{U}(x)} \bigwedge_{y \in V} f \sim_{\mathfrak{U}(y)} g$.

Especially the last interpretation shows how the axioms provide the means for the fundamental procedure of inference from a property in a point x to the corresponding local property (of a neighbourhood V).

Other concepts of topology find their natural expression in terms of the equivalence relations $\sim_{\mathfrak{U}(x)}$. The proofs and more about this subject can be found in the paper G.Aumann, Der abbildungstheoretische Zugang zur Topologie, Sitz.Ber. Bayer.Akad.d.Wissensch.Math.-Nat.Kl.1977.

T.M.K.DAVISON (Hamilton): The definition of quadratic forms.

Theorem. Let R be a commutative ring with identity. Suppose that either R has a pair of units u,v such that $u + v = 1$ or 2 is not a zero-divisor in R.
Let M be a free R-module. Then a function $f: M \to R$ is a quadratic form if, and only if,

$$f(\alpha x + \beta y) = (\alpha^2 + \alpha\beta)f(x) + (\alpha\beta + \beta^2)f(y) - \alpha\beta f(x - y),$$

for all $\alpha, \beta \in R$, all $x,y \in M$.

R.GER (Katowice): Some remarks on quadratic functionals.

Let X be a real linear space. A mapping $q: X \to \mathbb{R}$ is called a quadratic functional iff

(1) $q(x+y) + q(x-y) = 2q(x)+2q(y)$ for all $(x,y) \in X^2$.

Lemma 1. Suppose H to be a base of X considered as a linear space over the field $\mathbb{Q}$ of all rationals. Let $q_o: \frac{1}{2}(H+H) \to \mathbb{R}$ be an arbitrary function. Then there exists exactly one quadratic functional q on X such that $q|_{\frac{1}{2}(H+H)} = q_o$.

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Lemma 2. Let $q: X \to \mathbb{R}$ satisfy (1) and let $T \subset X$ be such that q is bounded on $\frac{1}{2}(T+T)$. Then q is bounded on $\mathbb{Q}(T)$ - the $\mathbb{Q}$-convex hull of T.

Now, assume that $(X,(\cdot|\cdot))$ is a real inner-product space and suppose that a quadratic functional $q: X \to \mathbb{R}$ satisfies a subsidiary condition

$$(2) \qquad \bigvee_{c \geq 0} \quad \bigwedge_{x,y \in X} \left(\|x\| = \|y\| \text{ implies } |q(x)-q(y)| \leq c\|x\|^2 \right).$$

It is known that, if q satisfies (1) and q is continuous, then q satisfies (2), but not conversely.

Theorem 1. Let $q: X \to \mathbb{R}$ satisfy (1) and (2) and let $T \subset X$ be such that the inner Lebesgue measure of $\{ \|x\| : x \in \mathbb{Q}(T)\}$ is positive. Then q is continuous provided q is bounded on $\frac{1}{2}(T+T)$.

The formula

$$(3) \qquad q(x) = \varphi(\|x\|^2), \quad x \in X$$

where $\varphi: \mathbb{R} \to \mathbb{R}$ is an additive function, produces quadratic functionals which satisfy (2). They do because their restrictions to an arbitrary sphere in X are constant. Conversely:

Theorem 2. If $\dim X \geq 2$ and if $q: X \to \mathbb{R}$ is a quadratic functional whose restrictions to each sphere in X are constant, then q is of the form (3).

There exist (discontinuous) quadratic functionals fulfilling (2) which are not of the form (3). However, they must be continuous on each sphere in X. This results immediately from the following

Theorem 3. If $q: X \to \mathbb{R}$ is a quadratic functional fulfilling (2) then

$$\bigwedge_{x,y \in X} \left(\|x\| = \|y\| \text{ implies } |q(x)-q(y)| \leq c \sqrt{\|x\|^4 - (x|y)^2} \right).$$

G.TARGONSKI (Marburg): Iteration theory and topological dynamics.

Z.Moszner (Demonstratio Math. 6 (1973), 309-327) has suggested, that "cross fertilization" should take place between various fields of application of the translation equation, such as

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continuous iteration, topological dynamics, geometrical objects, and so on. It is shown, that this idea is certainly fruitful as far as topological dynamics and continuous iteration is concerned. As examples, three things are mentioned. First, a theorem of V.I.Zidkov (Veh. Zap.Moskovsk.Univ. (Mat.) 163 (1952), 31-59) about embedding of natural iterates into a family of continuous iterates in a space of higher dimension. Second, a problem about a possible isomorphism of a family of continuous iterates to a family of differentiable iterates in the plane (Sibirskǐ, K.S., Introduction to topological dynamics, Noordhoff 1975, p.27). Third, it is pointed out that the more than 450 pages of A.Beck: Continuous flows in the plane, Springer 1974, can be considered as a handbook of the continuous solutions of the translation equation in the plane (with time as the variable in the semi-group); thus this book is very much relevant to continuous iteration of continuous mappings of the plane into itself.

J.MATKOWSKI (Bielsko-Biala): Integrable solutions of functional equations.

We consider nonlinear functional equations

(1) $\varphi(x) = h\left[x, \varphi[f(x)]\right]$ and (2) $\varphi[f(x)] = h[x, \varphi(x)]$,

where f and h are given and φ is unknown. Let $I = (0,1)$. We assume

(i) $f: I \to I$ is strictly increasing, f and f^{-1} are absolutely continuous in I and $f(I)$, respectively,

(ii) $h: I \times R \to R$, for each $y \in R$, $h(\cdot,y)$ is measurable and there is a $\eta: I \to R$ and a concave function $\gamma: [0,\infty) \to [0,\infty)$, $\gamma(t) < t$ for $t > 0$, such that $|h(x,y_1)-h(x,y_2)| \le \eta(x)\gamma(|y_1-y_2|)$ for $x \in I$, $y_1, y_2 \in R$.

Theorem 1. If (i) - (ii) are fulfilled, $h(\cdot,0) \in L(I)$ and $\eta \le f'$ a.e. in I, then there is a unique $\varphi \in L(I)$ which satisfies eq. (1).

Theorem 2. If (i) - (ii) are fulfilled, $0 < f(x) < x$ for $x \in I$, $h(f^{-1}(\ ),0) \in L[f(I)]$ and $\eta[f^{-1}(x)] < [f^{-1}(x)]'$ a.e. in I,

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then the general integrable solution of eq. (2) depends on an arbitrary integrable function.

E.SCHRÖDER (Hamburg): Zur Kennzeichnung der Lorentz-Transformationen.

Es seien (V,f) ein reeller Minkowski-Raum beliebiger Dimension ≥ 2 und d die zugehörige Distanzfunktion. Dann läßt sich die folgende Aussage (unter anderen) beweisen:
Ist $\varphi: V \to V$ eine surjektive Abbildung derart, daß die Bedingung

$$d(P,Q) = a \iff d(P^{\varphi},Q^{\varphi}) = a \qquad \forall P,Q \in V$$

für irgend ein festes $a \in \mathbb{R}$, $a < 0$, erfüllt ist, so ist φ eine Lorentz-Transformation.

Problems and Remarks

1. Remark. It would be desirable to develop a theory of "functional" equations where the unknown objects are formal power series rather than functions.

J.Aczél

2. Problem. Es sei K ein Schiefkörper. Man bestimme alle Lösungen $\varphi: K\backslash\{0\} \to K$ von

$$\varphi(uv) = u\varphi(v) + \varphi(u) \qquad (u,v \in K\backslash\{0\}) .$$

Insbesondere: gibt es Bijektionen[δ] $\varphi: K\backslash\{0\} \to K$, die dieser Gleichung genügen ? (Im Falle eines nichttrivialen Zentrums gibt es keine solchen Surjektionen und auch die allgemeine Lösung ist leicht zu bestimmen.)

G.Pickert

3. Problem. Let P_0^n be the algebra of polynomials (over R) of n variables, vanishing at 0. In P_0^1, the following is true. The Schröder equation $f\circ\omega = \lambda f$ has a non-trivial (not identically zero) solution only if ω is linear (this follows from the comparison of degrees). For $n > 1$, such a statement does not hold, as the following example, due to Kuczma, shows.

[δ] oder wenigstens Surjektionen

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Let ω: $x' = \mu x + P(x,y)$
$y' = \mu y - P(x,y)$

μ = const, P with $P(0,0) = 0$ an arbitrary polynomial. Then $f(x,y) = (x+y)^m$, $m \in \mathbb{N}$ arbitrary, is a solution of the Schröder equation with $\lambda = \mu^m$. The problem is to give a "natural" condition on ω, so that in that class the analogue of the P_0^1 statement holds: $f \circ \omega = \lambda f$ has a non-trivial solution only if ω is linear.

G.Targonski

4. Remark (answer to a problem posed by J.G.Dhombres at the 1975 meeting) and Problem.

Let G denote an abelian group which does not possess elements of order two. Consider the equation

(1) $F(x,F(y,z)) + F(x,F(z,y)) = F(F(x,y),z) + F(F(y,x),z)$ for all $x,y,z \in G$,

where $F: G \times G \to G$ is a function satisfying condition (a): $F(x,y) - F(x',y)$ depends only on $x'-x$ for all $y \in G$.

Theorem 1. The general solution of the equation (1) is given by

(2) $F(x,y) = \varphi(x) + \varphi(y) + c$ for all $x,y \in G$,

where a function $\varphi: G \to G$ satisfies Cauchy's equation

(3) $\varphi(x+y) = \varphi(x) + \varphi(y)$ for all $x,y \in G$

and the equation

(4) $\varphi(\varphi(x)) = \varphi(x)$ for all $x \in G$.

Solutions of the system of the equations (3) and (4) are characterized by the following

Theorem 2. A function $\varphi: G \to G$ satisfies the equations (3) and (4) iff there exist subgroups G_1, G_2 of G such that $G = G_1 + G_2$ and

$\varphi(x_1 + x_2) = x_1$ for all $x_1 \in G_1$, $x_2 \in G_2$.

The above theorems (and others concerning continuous solutions) give an answer to the question of J.G.Dhombres (cf. Aequationes Math. 14 (1976), p.230, and Glasnik Mat. 11 (31), 1976, p. 37-40).

The general solution of the equation (1) in an arbitrary abelian group is still unknown. It is easy to check that in an arbitrary

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abelian group every function of the form (2) satisfies the equation (1). But, e.g., in a Klein group there are solutions of the equation (1) which are not of the form (2).

J. Tabor

5. Remark (partial solution of Pickert's problem 2).
Let K be a division ring with more than two elements. We prove that there exists no bijection ϕ from $K\backslash\{0\}$ onto K which satisfies

$$\phi(uv) = u\phi(v) + \phi(u) \quad \text{for all } u,v \in K\backslash\{0\}.$$

Indeed, choose an arbitrary $u_o \in K\backslash\{0,1\}$ and denote

$$k = (u_o-1)^{-1}\phi(u_o), \text{ i.e., } \phi(u_o) = (u_o-1)k.$$

We prove our statement by proving that $(-k)$ is not in the range of ϕ. For, if there existed a $v_o \in K\backslash\{0\}$ with

$$\phi(v_o) = -k,$$

then

$$\phi(u_o v_o) = u_o\phi(v_o) + \phi(u_o) = -u_o k + (u_o-1)k = -k = \phi(v_o).$$

Because of the injectivity of ϕ, this would imply $u_o v_o = v_o$, i.e. $u_o = 1$ contrary to the choice of u_o.
Thus no such bijection, as required in problem 2, exists.

T.M.K. Davison - J. Aczél

6. Remark. Let us say that a strongly continuous one-parameter family $F(t)$ $(t \in \mathbb{R}^+)$ of contractive (bounded) operators has the semigroup property if $F(t+s) = F(t)F(s)$. In a paper yet available as preprint only, B. Misra and K. B. Sinha show the following. The inequality

$$\|[F(t+s) - F(t)F(s)]\psi\| = c_\psi t^\alpha s^\alpha$$

(where c_ψ is a constant depending on ψ only, $t \geq 0$, $s \geq 0$, $\alpha > 1$) $\psi \in L^2$ implies the semigroup property.

G. Targonski

7. Remark on Cayley sets. A Cayley set is the set of left (or right) inner translations of a semigroup. D. Zupnik has recently proved the following:
Theorem. Let F be a family of functions mapping a set S into itself. Suppose that F is closed under composition, and that

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there exists an element a of S such that $f_1(a) = f_2(a)$ implies $f_1 = f_2$ for all f_1, f_2 in F. Then F is a Cayley set if and only if there exists a function g mapping S into S such that (1) g commutes with every function in F, (2) the range of g coincides with the set $\{f(a) \mid f \in F\}$. This theorem extends the results announced by Zupnik at the 79th Summer Meeting of the Americal Mathematical Society, Kalamazoo, August 1975 (cf. Notices of the Amer.Math.Soc., 22 (1975), Abst. 726-39-17).

A.Sklar

8. Remarque au rapport de M.R.Beauvais pendant l'onzième colloque au sujet des équations fonctionnelles dans Oberwolfach en 1973 (Aequationes Math. 11, 1974, p. 281).

M.Beauvais a proposé de donner la solution générale de l'équation de translation

(1) $$U(U(x,\alpha_+),\beta_+) = U(x,\alpha_+ \circ \beta_+),$$

où $U: X \times G^+ \to X$, X est un ensemble arbitraire, G^+ est l'ensemble des éléments positifs d'un groupe $(G,\circ)$ totalement ordonné par une relation R, sous la forme

(2) $$U(x,\alpha\circ\beta^{-1}) = F(x,\alpha,\beta),$$

où $F: X \times R \to X$ est une solution de l'équation

(3) $$F(F(x,\alpha,\beta),\beta,\delta) = F(x,\alpha,\delta)$$

pour x de X. $(\alpha,\beta) \in R$ et $(\beta,\delta) \in R$, en utilisant un résultat de M. C.T.Ng au sujet de la solution générale de l'équation (3) dans le cas si R forme une relation fermée et transitive.

La reduction de l'équation (1) à l'équation (3) est simulée. En effet la formule (2) ne peut pas donner une solution générale de l'équation (1) puisque pour ce but il ne suffit pas connaitre toutes les solutions de l'équation (3), il faut connaitre toutes les solutions de (3) qui dépendent seulement de l'expression $\alpha\circ\beta^{-1}$, c'est-à-dire les fonctions pour lesquelles

(4) $$F(x,\alpha,\beta) = F(x,\alpha\circ\beta^{-1},e)$$

pour $(\alpha,\beta) \in R$, où e est l'élément neutre du groupe G. En effet il suffit pour recevoir (4) de poser dans (2) $\alpha\circ\beta^{-1}$ au lieu de α et e au lieu de β. Mais d'après (3) la fonction $F(x,\alpha,e)$ doit remplir (1) donc pour connaitre toutes

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les solutions de (3) remplissants (4) il faut connaitre toutes les solutions de (1).

Remarquons qu'il existe des solutions de (3) qui n'ont pas de la propriêté (4). Par exemple telle est la fonction

$$F(x,\alpha,\beta) = \begin{cases} x & \text{pour } 0 \le \alpha < \beta \text{ et} \\ 0 & \text{pour } \beta > \alpha < 0 , \end{cases}$$

où G est le groupe additif des nombres rêels et $X = G$.

Z.Moszner

9. Remark on a Theorem of M.J.Frank concerning Simultaneous Associativity of Functions. Let B denote the set of functions $F: [0,1]^2 \to [0,1]$ such that

$F(0,x) = F(x,0) = 0$, $F(x,1) = F(1,x) = x$ for all $x \in [0,1]$.

Given F in B, define $F^\wedge$ by: $F^\wedge(x,y) = x + y - F(x,y)$.

M.J.Frank has proved the following:

Theorem. The set of continuous F in B such that both F and $F^\wedge$ are associative is the set $\{F_s \mid 0 \le s \le \infty\}$, and all ordinal sums of the F_s's, where

$$F_0(x,y) = \min (x,y),$$
$$F_1(x,y) = \operatorname{Prod} (x,y) = x \cdot y,$$
$$F_\infty(x,y) = \max (x+y-1,0),$$
$$F_s(x,y) = \frac{\log \left[1 + \frac{(s^x-1)(s^y-1)}{s-1}\right]}{\log s} \quad \text{for } 0 < s < \infty,\ s \neq 1 .$$

This theorem completely settles Part I of Problem P 126, Aequationes Math. 11, 1974, 312-313.

A.Sklar

10. Problem. Consider a vector functional equation

(1) $$A \cdot m(t) \cdot \underline{y}(h(t)) = \underline{y}(t),$$

where A is an n by n regular constant matrix, $n \ge 2$, $m \in C^n(I)$, I being an open (bounded or unbounded) interval of $\mathbb{R}$, $m(t) \neq 0$ for all $t \in I$,

$\underline{y}: I \to \mathbb{R}^n$, $\underline{y} \in C^n(I)$ with a non-zero Wronskian on I,

h being a bijection of I onto I of the class $C^n(I)$ with $dh(t)/dt \neq 0$ on I.

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Then writing $\alpha(\underline{y})$ instead of the left side of (1), all the α's that for a given $\underline{y}$ satisfy (1) form a group G. "Characterize $\underline{y}$ with respect to the G."
For the partial answer for cyclic groups G, see
F.Neuman: On solutions of the vector functional equation $\underline{y}(\xi(x)) = f(x)\cdot A\cdot\underline{y}(x)$, to appear in Aequationes Math. 15 (1977), No 2-3.

F.Neuman

11. Problem. Let $(X,(\cdot|\cdot))$ be a real inner-product space and let $V: X \to X$ be an additive mapping. Put

$$q(x) := (V(x)|x)\ ,\ x \in X,$$

and assume that

$$\bigvee_{c\geq 0}\ \bigwedge_{x,y\in X} (\ \|x\| = \|y\| \text{ implies } |q(x)-q(y)| \leq c\ \|x\|^2\ )\ .$$

It is known that q is continuous provided V is linear.
Is the converse true ?

R.Ger

F.Neuman (Brno)

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