MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 5 / 1979

# " BOOLESCHE ALGEBREN "

28.1. bis 3.2.1979

Leitung der Tagung: Sabine Koppelberg, Berlin
J. Donald Monk, Boulder

Die Tagung über Boolesche Algebren verfolgte in erster Linie zwei Ziele.

Zum einen wurde gemeinsam die kürzlich erschienene Arbeit von Jussi Ketonen : " The Structure of Countable Boolean Algebras " (Ann.Math. 108 (1978)) erarbeitet. Die Arbeit wurde dazu in einzelne Referate aufgegliedert und an den Vormittagen vorgetragen und diskutiert.

Zum anderen wurde an den Nachmittagen ein Überblick über die neuesten Forschungsergebnisse in den verschiedensten Gebieten innerhalb der Theorie der Booleschen Algebren gegeben.

Ein weiteres Ergebnis der Tagung bestand darin, daß S. Shelah zusammen mit anderen Teilnehmern 27 von 64 offenen Problemen, die im Preprint - Eric K. van Douwen, J. Donald Monk, Matatyahu Rubin: " Some Questions about Boolean Algebras " (Zürich, Dez. 1978) - erwähnt werden, ganz oder teilweise löste.

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## VORTRAGSAUSZÜGE

### " Nowhere dense rigid Boolean Algebras "

A Boolean Algebra B is said to be nowhere dense rigid, if for ervery endomorphism f of B the ideal $I_f$ generated by $\{x / fx = 0\} \cup \{x / fy = y \text{ for } y \leq x\}$ is dense in B.

If B is atomless, then B is mono-rigid (all one-to-one endomorphisms are trivially the identity).

Theorem 1: (GCH) For every $\lambda^+ = 2^\lambda$ there are $2^{\lambda^+}$ atomless nowhere dense rigid Boolean Algebras of cardinality $\lambda^+$.

Theorem 2: (GCH) For every Boolean Algebra B there is an atomless nowhere dense rigid Boolean Algebra $B^*$, such that B is embeddable into $B^*$ and B is a quotient algebra of $B^*$.

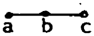
R. Bonnet, Villeurbanne

"Boolean Algebras as subalgebras of ortomodular lattices"

The loop-lemma and the bundle-lemma have been presented in the following form:

Loop-lemma: In an orthomodular lattice whose maximal Boolean subalgebras (blocks) have at most $2^3$ elements the Greechie-graph does not contain a loop of order 3 or 4.

 loop of order 3  loop of order 4

Here 

represents a block of L with the atoms a, b, c.

Bundle-lemma: Let $\{B_i\}_{i\in I}$ be a set of Boolean algebras such that the following conditions hold,

(a) $B_i \cap B_j$ is a subalgebra of $B_i$ and $B_j$ with the same induced structures,

(b) if $x, y \in B_i$ ; $y, z \in B_j$ and $x \leq_i y$, $y \leq_j z$ then there exists $k \in I$ with $x, z \in B_k$ and $x \leq_k z$,

(c) if $x \in B_i$ and $y \in B_j$ there exists $z \in B_k$ for some $k \in I$ such that

(c1) if $i = j$ then $k = i$,

(c2) there exist $l, m \in I$ with $x, z \in B_l$, $y, z \in B_m$, $x \leq_l z$ and $y \leq_l z$,

(c3) if $x, u \in B_r$, $y, u \in B_s$ and $x \leq_r u$, $y \leq_s u$ then there exists $t \in I$ with $z, u \in B_t$ and $z \leq_t u$.

Define on $L = \bigcup_{i\in I} B_i$ the relation $a \leq b$ iff there exists $i \in I$ with $a, b \in B_i$ and $a \leq_i b$ and the map $': L - L$ by $a' = a'^{i}$ for $a \in B_i$. Then $\leq$ provides L with a lattice structure such that L together with ' is an orthomodular lattice.
Conversely, starting out with an orthomodular lattice L the set of blocks $\{B_i\}_{i\in I}$ of L have the properties of the bundle-lemma.

A decomposition of an orthomodular lattice L was mentioned:
if $c \in C(L) = \bigcap \{B \subseteq L \,/\, B \text{ block}\}$ then L is isomorphic to $[0,c] \times [0,c']$.
Here the fact is used that an interval of an orthomodular lattice is (naturally) again an orthomodular lattice.
Some problems have been stated: better structural or decomposition theorems for orthomodular lattices would be useful.

[1] R. Greechie: Orthomodular lattices admitting no states
J. Combin. Theory 10 (1971), 119-132

[2] S.S. Holland jr.: A Radon-Nikodym theorem in dimension lattices
Trans. AMS 108 (1963), 66-87

[3] G. Kalmbach: Orthomodular lattices do not satisfy any special lattice equation
Arch. d. Math. 28 (1977), 7-8

G. Kalmbach, Ulm

## " Isomorhism types of complete Boolean Algebras "

For a complete Boolean algebra (cBA) B, let $T(B) = \{\tau(B/b) / b \in B\}$ where $\tau(B/b)$ is the isomorphism type of the relative algebra B/b of B. Define a partial order on T(B) by $s \leq t$ iff $s = \tau(A_0)$, $t = \tau(A)$ and $A \cong A_0 \times A_1$ for some cBA $A_1$. T(B) has a largest and a smallest element.

Theorem A: $(T(B), \leq)$ is isomorphic to the structure of global sections of S, where S is a Hausdorff sheaf of linear orders over X, X the Stone space of $R(B) = \{x \in B / f(x) = x$ for ervery f Aut B$\}$.

Theorem B: Both $(T(B), \leq)$ and $(T(B), \geq)$ are distributive lattices, Stone algebras and Heyting algebras satisfying $(a \to b) \vee (b \to a) = 1$ for $a,b \in T(B)$ (For many cBA's B, $(T(B), \leq)$ is a complete lattice.)

The following is proved in a paper by Grigorieff as being a result of Solovay but seems to be unpublished. Call B weakly homogenous iff $R(B) = \{0, 1\}$ (iff $(T(B), \leq)$ is a linear order).

Theorem C: B is a weakly homogenous cBA iff B is a power of a homogenous cBA.

There are two consequences of Theorem C:

Theorem D: Let B be any cBA; let C be the regular open algebra of $X^\lambda$ where $\lambda \geq \omega$ and X is the Stone space of B. Then B is completely embeddable into C, C is homogeneous and, if c(B) (cellularity of B) $\leq \alpha^+$ for some $\alpha$, then $c(C) \leq (2^\alpha)^+$.

Theorem E: Let B be a cBA s.t. Aut B is infinite then $|\mathrm{Aut}\, B|^\omega = |\mathrm{Aut}\, B|$.

Sabine Koppelberg, FU Berlin (z.Zt. ETH Zürich)

## " On Hopfian Boolean Algebras "

A Boolean algebra is Hopfian if every onto endomorphism is one-to-one, and onto-rigid if the identity is the only non-trivial onto endomorphism.

Theorem: (MA + ¬CH) There are no Hopfian Boolean algebras with infinitely many atoms which are of power $< 2^\omega$.

Corollary: In particular, no countable Boolean algebra is Hopfian.

Theorem: (with M. Rubin) For every uncountable $\kappa$, there is an onto-rigid Boolean Algebra of power $\kappa$.

Detailed prooofs of these results will appear in Proceedings of the American Mathematical Society.

James T. Loats, Lawrence

"An algebraic approach to stability in model theory"

Starting from papers by Shelah [3,4] and Day [1], we define some concepts of "rank" for Boolean algebras trying to characterize classes of Boolean algebras having some interest in model theory. (in the sense of Keisler's Problem 6 [2])

Some results are:

(a) Algebraic characterization of $\omega$-stable theories (via superatomic Boolean algebras)

(b) New proof of $\aleph_1$-categoricity of strongly minimal theories (via principal Boolean algebras)

(c) Approach to superstability (via k-superatomic Boolean algebras).

[1] G.W. Day: Pac.J. of Math. 23 (1967) 479-489

[2] H.J. Keisler: J.Austral.Math.Soc. (Series A) 21 (1976) 257-266

[3] S. Shelah: J. Symbolic Logik 37 (1972) 107-113

[4] " : Logique et Analyse, Nouvelle serie $18^{o}$ annee (1975) 241-307.

A. Marcja, Florenz

"Depth and length of Boolean algebras"

Let depth A = sup $\{|X| \,/\, X \subseteq A,$ X well-ordered$\}$

Theorem 1: If depth A is singular with cofinality $\omega$ then depth A is attained. There are counterexamples otherwise.

Theorem 2: Suppose $cf\kappa > \omega$, A has no chain of type $\kappa$, B has no chain of type $cf\kappa$ then A * B has no chain of type $\kappa$.

There are counterexamples otherwise.

So we obtain a complete description of depth, and attainment of it, for free products with arbitrarily many factors. One can easily describe the depth of products.

$P_\omega *_A P_\omega$ has depth $2^{\aleph_0}$, where A is the finite-cofinite algebra.

Theorem 3: (GCH) Let $|B| = \kappa^+$, $\lambda \leq$ depth B, $\aleph_0 \leq \lambda \leq \mu \leq \kappa$ then there is $A \subseteq B$ so that $|A| = \mu$ and depth A $= \lambda$.

Under GCH there exists a Boolean algebra of power $\kappa^{++}$ with depth $\kappa^+$ and every subalgebra of power $\kappa^{++}$ has depth $\kappa^+$.

<u>Theorem 4:</u> Suppose $\lambda \leq \kappa^+$ then there exists a rigid Boolean algebra of power $\kappa^+$ and depth $\lambda$.

Let length A = sup $\{ X \;/\; X \subseteq A,\ X$ simply ordered $\}$ length $\leq 2^{\text{depth}}$.

The analog of Theorem 1 holds.

<u>Theorem 5:</u> If $\kappa > \aleph_0$ is regular and A and B have no chains of power $\kappa$, then A * B has no chain of power $\kappa$,

J. Donald Monk, Boulder

<u>" Cardinal Invariants on Boolean Algebras "</u>

length (B) = sup $\{ |C| \;/\; C \subset B$ is a chain $\}$

depth (B) = sup $\{ |C| \;/\; C \subset B$ is well-ordered $\}$

width (B) = sup $\{ |A| \;/\; A \subset B$ is (pairwise) disjoint $\}$

spread (B) = sup $\{ |A| \;/\; A \subset B$, A generates an ideal minimally $\}$
(i.e. if $a \in A$, $A - \{a\}$ does not generate the same ideal in A )

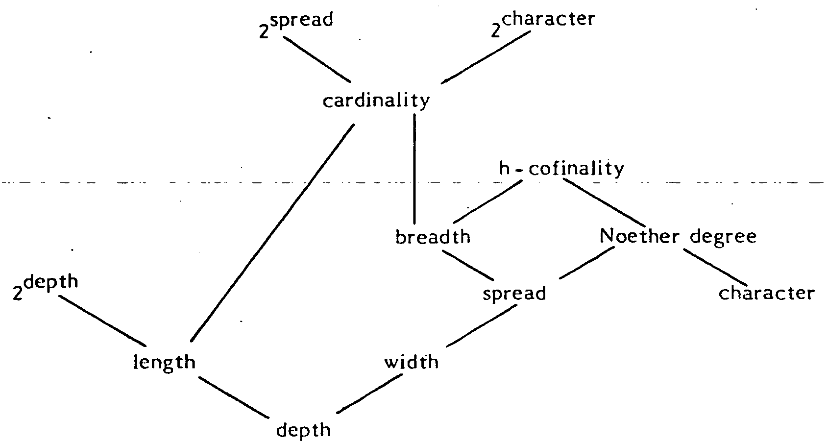
breadth (B) = sup $\{ |A| \;/\; A \subset B$ is an antichain ( pairwise incomparable ) $\}$

h-confinality (B) = min $\{ \alpha \;/$ every subset of B has a cofinal subset of cardinality $\leq \alpha \}$

Noether degree (B) = min $\{ \alpha \;/$ every ideal of B can be generated by $\leq \alpha$ elements $\}$

character (B) = min $\{ \alpha \;/$ every maximal ideal of B can be generated by $\leq \alpha$ elements $\}$

No other inequalities besides those depicted below are provable in ZFC.




P. J. Nyikos, Auburn

" On the reconstruction of Boolean algebras from their automorphism groups "

A Boolean algebra B is called faithful, if for every direct summand $B_1$ of B: if $B_1$ is rigid (that is, it does not have any automorhisms other then the identity) then there is $B_2$ such that $B = B_1 \times B_1 \times B_1 \times B_2$.
Let B be a complete Boolean algebra, then B can be uniquely represented as $B = B^R \times B^D \times B^D \times B^F$, where $B^R, B^D, B^F$ are pairwise totally different (that is, no two of them have non-zero isomorphic direct summands) $B^R, B^D$ are rigid and $B^F$ is faithful. Aut(B) denotes the automorphism group of B.

Theorem 1: (a) Let $B_1, B_2$ be complete Boolean algebras, then $\mathrm{Aut}(B_1) \cong \mathrm{Aut}(B_2)$ iff $B_1^F \cong B_2^F$ and $B_1^D$, $B_2^D$ have the same cardinality.
(b) Moreover, if $B_1, B_2$ are faithful Boolean algebras and $\mathrm{Aut}(B_1) \cong \mathrm{Aut}(B_2)$ then the completion of $B_1$ is isomorphic to the completion of $B_2$.

Matatyahu Rubin, Beer Sheva

" Constructing examples by imitating forcing "

We shall define below what is a $\lambda^+$-simple order P and what are appropriate functions for it.
The main result is:

Theorem: Suppose $\lambda = \aleph_0, \mathfrak{Q}_\lambda$ or $\lambda$ strongly inaccessible.
If P is $\lambda^+$-simple, $F^\alpha (\alpha<\lambda)$ are appropriate, then there is a directed $G \subseteq P$ which is $F^\alpha$-generic for every $\alpha$, which means:
for every $\bar{y}, \tau(\bar{x}) \in G$, $\bar{y} \leq x$ there is $\tau'(\bar{x})$, $\tau(\bar{x}) \leq \tau'(x') \leq F_{\bar{y}}(\tau'(\bar{x}')) \in G$;
also for every $\bar{y}$ for some $\tau, \tau(\bar{y}) \in G$

Def.: A partial order P is $\lambda^+$-simple provided that:
(1) its elements have the form $\tau(\bar{x}), \tau \in T$, $\bar{x}$ (always) an increasing sequence of ordinals $< \lambda^+$ of length $\alpha_\tau < \lambda$
(2) $\tau^1(\bar{x}^1) \leq \tau^2(x^2)$ depend only on $\tau^1, \tau^2$ and the order relation between the $x_i^1$'s and $x_j^2$'s
(3) for every $\beta < \alpha_\tau$, $\tau(\bar{x})/\beta = (\tau/\beta)$ $\langle x_i : i<\beta \rangle \in P$ is increasing and continous in $\beta$, and if $\tau(\bar{x})/\beta \leq \sigma(\bar{y})$ $y_i < x_\beta$ for every $i < \alpha_\sigma$ then $\tau(\bar{x}), \sigma(\bar{y})$ are compatible
(4) Every increasing continous chain has a least uppper bound.

Def.: F is appropriate, if for every $\bar{y} \leq \bar{x}$, $F_{\bar{y}}(\bar{\tau}(\bar{x}))$ belongs to $\Gamma_{\leq}$, and has the force $\tau'(\bar{x})$. $\tau'$ depends on $\bar{\tau}$ and the equalities between the $x_i$'s and $y_j$'s and $\tau(\bar{x}) \leq \tau'(x)$.

S. Shelah, Jerusalem

" Boolean algebras without rigid or homogeneous factors "

We say that a Boolean algebra has no rigid or homogeneous factors if for every nonzero element u of B, the partial algebra $B \restriction u$ is neither rigid nor homogenous The existence of such algebras was proved by S. Koppelberg assuming $\Diamond$ and by Balcar and the author without this assumption. We shall give some more results about these algebras.

Theorem 1: Given a regular uncountable cardinal $\kappa$, there is a Boolean algebra B of power $\kappa$, satisfying CCC and with no rigid or homogeneous factors. The completion $B^c$ of B has the same property, hence we may asssume that B is complete whenever $\kappa^{\aleph_0} = \kappa$.

Theorem 2: For ervery uncountable regular cardinal $\kappa$ there exists a family $B_\alpha$, $\alpha < 2^\kappa$ of CCC Boolean algebras of power $\kappa$, such that for every $\alpha, \beta$ we have

(i) $B_\alpha$ and $B_\alpha^c$ have no rigid nor homogeneous factors

(ii) $B_\alpha^c$ is not isomorphic to $B_\beta^c$ whenerver $\alpha \neq \beta$.

Hence if $\kappa^{\aleph_0} = \kappa$, we may assume that all $B_\alpha$'s are complete.

Theorem 3: For every singular cardinal $\kappa$, there are $2^\kappa$ non isomorphic Boolean algebras of power $\kappa$ without nonzero rigid or homogeneous factors.

Theorem 4: Every Boolean agebra B can be completely embedded in a complete Boolean algebra C with no rigid or homogeneous factors such that C and B have the same saturation whenever B is infinite.

It is known that if we want to prove a theorem similar to Theorem 4 the Boolean algebra C must satisfy some restrictions on automorphisms. It turns out that there is a simple condition (P) such that the statement " every Boolean algebra can be embedded in a Boolean algebra satisfying (P) and having the same saturation " is a theorem of ZFC but the same statement with (P) replaced by $(\neg P)$ is not provable in ZFC. The condition (P) can be formulated as follows:

(P) if $B \upharpoonright u$ is homogeneous then u is an atom of B or in other words, B can have at most one atom and the atomless part of B has no homogenous factors.

I would like to mention some recent results of Balcar, Franěk and Trnková. Let us call a Boolean algebra semifree if it contains an independent subset of the same power. Monk has shown that assuming GCH, every infinite complete Boolean algebra is semifree.

Theorem * (Balcar and Franěk) Every infinite complete Boolean algebra is semifree.

Let us consider the cube problem for free products of Boolean algebras instead of product. Trnková and Koubek (1977) constructed a Boolean algebra B isomorphic to the free product B + B + B but not to the free product B + B of two copies of B. This algebra was not countable.

Theorem (Trnková) There are no countable Boolean algebras B such that B is isomorphic to B + B + B but not to B + B.

* We have the following corollary to the Theorem of Balcar and Franěk.

Corollary: For every infinite complete Boolean algebra B there are $2^B$ ultrafilters on B.

Petr Štěpánek, Praha

" The structure of countable Boolean algebra " (Jussi Ketonen, Ann.Math. 108)

Das Tarskische Würfelproblem wird gelöst: Es gibt eine abzählbare Boolesche Algebra A mit $A = A \times A \times A$, $A \neq A \times A$.

Das heißt, man kann die Gruppe $\mathbb{Z}/_{2\mathbb{Z}}$ in die Halbgruppe (BA, x) der Isomorphietypen von abzählbaren Booleschen Algebren einbetten. Ketonen zeigt sogar, daß sich jede abzählbare Halbgruppe S in (BA, x) einbetten läßt. Dazu ordnet er jeder abzählbaren Booleschen Algebra ein Invariantensystem zu, das die Boolesche Algebra bis auf Isomorphie eindeutig bestimmt. Dann konstruiert er für jedes $s \in S$ eine Boolesche Algebra $A_s$ durch Angabe ihrer Invarianten.

Im einzelnen:

Kapitel 0,1,2 (Vortragende: U.Brehm Freiburg, K. Potthoff, Kiel)

Der Rang r(B) einer Booleschen Algebra B ist die Länge der Folge der Cantor-Bendixson Ableitung $(X^{\alpha})_{\alpha < r(B)}$ des Stone - Raumes X von B. B ist uniform, wenn $r(X) = r(X - D)$ für alle abzählbaren clopen $D \subset X$. Sei $P(X) := \bigcap_{\alpha} X^{\alpha}$ der perfekte Kern von X. Für clopen $A \subset X$ hängt r(A) - X uniform - nur von $r(A \cap P(X))$ ab. $r_X$ sei die dadurch bestimmte Funktion auf der zu P(X) dualen

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Booleschen Algebra. $\Lambda$ sie die Boolesche Algebra mit abzählbar vielen Erzeugenden. Wir können für abzählbares X, $r_X$ als Funktion $r_X : \Lambda \to \omega_1$ auffassen.

<u>Satz:</u> (a) Jede abzählbare Boolesche Algebra B zerlegt sich in C x D, C uniform, B superatomar. C und D bis auf Isomorphie eindeutig.

(b) Die Isomorphietypen der abzählbaren uniformen Booleschen Algebra B entsprechen via $B \mapsto r_X$ eindeutig den Isomorphietypen von "Rang-Funktionen" $r : \Lambda \to \omega_1$ $(r(a \dot{\cup} b) = \max(r(a), r(b))$.

<u>Kapitel 3 (Vortragende: S. Kemmerich, M. M. Richter , Aachen)</u>

$r : \Lambda \to \omega_1$ sei eine Rangfunktion. Wir geben eine Folge von Invarianten an:

Sei $b \in \Lambda$: $inv_0(b) = r(b)$

$inv_{n+1}(b) = \{(inv_n(c_1), \ldots, inv_n(c_k)) / c_1 \dot{\cup} \ldots \dot{\cup} c_k = b\}$

Man definiert sinnvoll $inv_\alpha(b)$ für alle $\alpha < \omega_1$.

Für $\beta \geq$ Tiefe $(\Lambda, r)$ ist $inv_\beta(b)$ schon durch $inv_{\text{Tiefe}(\Lambda, r)}(b)$ bestimmt.

<u>Satz:</u> $(\Lambda, r)$ ist bis auf Isomorphie durch $inv_\alpha(1_\Lambda)$, $\alpha =$ Tiefe $(\Lambda, r)$ bestimmt.

Die Mengen der Form $inv_\alpha(1_\Lambda)$ und die Mengen der Form $inv_\alpha(1_\Lambda)$ für $\alpha \geq$ Tiefe $(\Lambda, r)$ lassen sich intern charakterisieren.

<u>Kapitel 4 (Vortragender: P. Schmitt, Heidelberg)</u>

Die Struktur der Mengen $inv_n(1_\Lambda)$, $n = 0, 1, 2, 3$ wird genauer untersucht. Es ergeben sich hier besonders einfache Beschreibungen. Zum Beispiel sind die Mengen $inv_1(1_\Lambda)$ bijektiv bezogen auf bestimmte Funktioen $f : \omega_1 \to \omega \cup \{\infty\}$. Das alles sind technische Vorbereitungen für Kapitel 5-7.

<u>Kapitel 5, 6, 7 ( Vortragender: K.-P. Podewski, Hannover)</u>

Dieses Kapitel ist eigentlich das Kernstück der Arbeit.

Eine Halbgruppe S wird in eine Menge von Invarianten (der Stufe 4) eingebettet.

Trotz einiger Verbesserungen, die der Vortragende vornehmen konnte, bleibt die Konstruktion sehr unübersichtlich und technisch. Vielleicht findet sich ein einsichtigerer Beweis.

Martin Ziegler, Berlin

S. Kemmerich (Aachen)