MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 24|1979

Hyperbolische Geometrie

3.6. bis 9.6.1979

Die Tagung wurde von Herrn W. Klingenberg (Bonn) organisiert und geleitet. Eine große Zahl deutscher und ausländischer Mathematiker war nach Oberwolfach gekommen, um über neue Forschungsergebnisse zu berichten und über aktuelle Probleme zu diskutieren.

Die Hauptthemen sowohl in den Diskussionen als auch in den Vorträgen waren: Riemannsche Flächen, das Eigenwertspektrum Riemannscher Flächen, 3-Mannigfaltigkeiten konstanter negativer Krümmung, negativ und nichtpositiv gekrümmte Mannigfaltigkeiten.

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Vortragsauszüge

W. Ballmann:

Flat Halfplanes

If M is a manifold of nonpositive curvature and c a closed geodesic in M, then any lift of c to the universal cover H of M is invariant under some axial isometry of $\pi_1(M)$. If the curvature of M is negative, then axial isometries satisfy certain conditions, which enable one to draw conclusions on $\pi_1(M)$ etc. In the general case of nonpositive curvature it turns out, that axial isometries satisfy these conditions if the underlying closed geodesic is not boundary of a flat half plane. Here a geodesic g is said to be boundary of a flat half plane, if there exists a totally geodesic, isometric immersion $i : [0,\infty) \times R \longrightarrow M$ s.t. $i(0,t) = g(t)$. One gets results of the following type:

<u>Theorem</u>. Let M be of nonpositive curvature, $vol(M) < \infty$. Suppose M admits a geodesic (not necessarily closed) which is not boundary of a flat half plane. Then $\pi_1(M)$ has free (non abelian) subgroups, the geodesic flow has a dense orbit in T_1M, there exists ∞-many closed geodesics in M.

In the proof one mainly has to establish the existence of a closed geodesic, which is not boundary of a flat half plane. There are also various other applications.

P. Buser:

On Bers' Constant L_g

Let S be a compact Riemann surface of genus $g \geq 2$ endowed with the Poincaré metric. A set of $3g-3$ mutually disjoint simple closed geodesics $\gamma_1, \ldots, \gamma_{3g-3}$ is called a partition. Bers' constant is defined by

$$L_g = \inf\{\ell \mid \text{every } S \text{ has a partition with length } (\gamma_i) \leq \ell\}.$$

Bers proved by a compactness argument $L_g < \infty$. Using hyperbolic geometry and volume estimates for cylinders it was shown:

<u>Theorem 1</u>. Each S has a partition with $\ell(\gamma_i) \leq 2(i+1)\log 4\pi(g-1)$, $i = 1, \ldots, 3g-3$. Three applications were given. The first is on the eigenvalue $\lambda_i(S)$.

<u>Theorem 2</u>. $\lambda_{2g-2}(S) > \frac{1}{2}(2g)^{-6g}$ (R. Schoen, S. Wolpert).

Note that for any $\varepsilon > 0$ there exist S s.t. $\lambda_{2g-3} < \varepsilon$.

Also a rough fundamental domain B for Teichmüller space was given und the following was shown.

<u>Theorem 3</u>. Each S occurs at most $\exp(64g^2)$ times in B.

<u>Theorem 4</u>. If $\mathrm{Evsp}(S_1) = \ldots = \mathrm{Evsp}(S_n)$, then $n \leq \exp(12g^3)$. Compare Wolpert's talk. Finally the following estimate was given:

<u>Theorem 5</u>. $-2 + \sqrt{6g-3} \leq L_g \leq (6g-4)\log 4\pi g$

P. Eberlein:

Isometry Groups of Simply Connected Manifolds of Nonpositive Curvature

Let H be a complete, simply connected manifold of nonpositive sectional curvature. A subgroup Γ of the isometry group I(H) satisfies the duality condition (DC) if for every geodesic γ of H there are $\varphi_n \in \Gamma$ such that $\varphi_n p \longrightarrow \gamma(\infty)$, $\varphi_n^{-1}(p) \longrightarrow \gamma(-\infty)$. If $M = H/\Gamma$ is smooth, then Γ satisfies DC if every $v \in T_1M$ is nonwandering with respect to the geodesic flow.

DC imposes strong conditions on Γ. If Γ satisfies DC and $A \triangleleft \Gamma$, then H has a Euclidean factor on which A acts by translations. If I(H) satisfies DC and $I_0(H) \neq \{\ell\}$, then some orbit of $I_0(H)$ is a noncompact symmetric space. Hence if H is homogeneous, I(H) satisfies DC if H is symmetric.

If I(H) is noncompact and acts minimally on $H(\infty)$, then I(H) satisfies DC and either I(H) is discrete, $H = R^n$, or H is a rank 1 symmetric space. Conversely if I(H) satisfies DC and some geodesic does not bound a flat half plane, then I(H) acts minimally on $H(\infty)$.

If H/Γ is smooth and of finite volume, then Γ satisfies DC. If H has a nonsymmetric de Rham factor that admits a geodesic not bounding a flat half plane, then H/Γ has a finite cover by a Riemannian product.

W. Fenchel :

On Trignometry in Hyperbolic 3-Space

In 1891 F. Schilling discovered that spherical trignometry with complex arguments admits a useful intepretation in the line geometry of hyperbolic 3-space H^3. A new approach to this, avoiding some short-comings of the older ones, can be based on the following observation. Consider the conformal model of H^3 in the upper half-space. The two points of infinity of a line in H^3 are the fixed points of a Möbius involution of the extended complex plane. This involution is determined by either one of two opposite 2x2 matrices with trace 0 and determinant 1. It is possible, by a consistent convention, to associate each of these matrices with one of the orientations of the line. The trignometrical theorem for hexagons with all angles right can then be obtained as relations between the traces of products of such "line matrices".

H.C. Im Hof :

The Flow of Weyl Chambers

Let X be a symmetric space of noncompact type. Choose $p \in X$, a flat F through p, and a Weyl chamber C in F based at p. These choices give rist to an Iwasawa decomposition of $G = I(X)$, namely $G = KAN$. If $M = Z_K(A)$ is the centralizer of A in K, then we have a bundle $Q = G/_M \longrightarrow G/_K = X$. Q is the space of all Weyl chambers in X. $\varphi_a(gM) = gaM$ $\underline{a}$ defines an action of A on Q.

<u>Theorem</u>. This action is an Anosov action.

Proof. The differential of φ_a is expressed in terms of the adjoint representation of a on g. The root space decomposition of (g,a) and the ordering of the roots induced by C give rise to a decomposition $g = m \oplus a \oplus n_+ \oplus n_-$. Hence $T_C Q = g/_m = a \oplus n_+ \oplus n_-$. This is the desired Anosov splitting of $T_C Q$.

A. Marden :

Kleinian Groups and Compactification of Moduli Space of Riemannian Surfaces

A. Marden gave an exposition of joint work with C. Earle concerning the analytic compactification of the moduli space of closed Riemannian surfaces of genus $g \geq 2$ and some new complex parameters for the Teichmüller space T_g. These parameters arise from an explicit geometric construction intrinsic in the surfaces. The proof of the theorem involves Kleinian group theory, more precisely the consideration of quasifuchsian space and its boundary. The analytic parameters arise from a comparison of the algebraic and appropriate geometric limit of certain cusps on the boundary. A discussion of the applications of the general result to an explicit example was given.

B. Maskit :

Isomorphisms between Kleinian groups

The classical isomorphism theorem of Fenchel and Nielsen gives necessary and sufficient conditions for an isomorphism

between finitely generated Fuchsian groups to be geometric. We discuss the problem of generalizing this theorem to the case of Kleinian groups. Let G, G^* be Kleinian groups and let $\varphi : G \longrightarrow G^*$ be an isomorphism. Our problem is to give conditions on the groups G and G^*, and on the isomorphism φ so that there exists a homeomorphism f with $f \gamma f^{-1} = \varphi(\gamma)$, for all $\gamma \in G$, (we ask either that f map the closed 3-ball onto itself, or that f map the set of discontinuity $(\subset S^2)$ of G onto that of G^*. We construct several examples, and use these examples to write down some conditions (such as: G is geometrically finite, φ and φ^{-1} preserve parabolic elements, φ and φ^{-1} preserve boundary elements,...). We then state two theorems giving sufficient conditions (Maskit, J. d'Analyse Math. 1977; A. Marden + B. Maskit, Invent.Math. 1979)

G.D. Mostow:

A New Class of Negatively Curved Algebraic Surfaces

A new class of subgroups $\Gamma \subset U(2,1)$ generated by complex reflections and depending on an integer p, $3 \leq p \leq 5$, and a complex number φ of modulus 1 and $\left|\frac{\arg \varphi^3}{\pi}\right| \leq \frac{1}{3}\left(\frac{\pi}{2} - \frac{\pi}{p}\right)$ was defined. An outline of the proof of the following result was given:

For at most a finite number of φ, Γ is a discrete subgroup of $U(2,1)$. If $\frac{\arg}{\pi} \in \mathbb{Q}$, there is a complex analytic manifold Y on which Γ operates discontinuously. For any torsion

free subgroup Γ_0 of finite index in Γ, $M = \Gamma_0 \backslash Y$ is a compact Kaehler manifold which is an algebraic surface. For an infinite number of φ, M carries a Kaehler metric of negative sectional curvature whose simply connected cover is not the ball.

Together with Y.T. Sin, the ratio $\frac{c_1^2}{c_2}$ has been computed for such surfaces. For example, if $p = 5$, $\arg \varphi = \frac{\pi}{20}$, then $\frac{c_1^2}{c^2} = \frac{852}{298} < 3$.

S. Murakami:

Harmonic Forms with Values in a Complex Line Bundle over a Complex Torus

Let T be a complex torus and F be a line bundle over T. Let $\underline{F}$ be the sheaf of germs of holomorphic sections of F, and let $H^q(T, \underline{F})$ be the q-th cohomology group of T with coefficients in $\underline{F}$. It was shown that the group $H^q(T, \underline{F})$ can be completely determined by analysing harmonic forms over T with values in F, thus giving an analytical proof to the results in algebraic geometry due to Mumford and Kempf. The method is a generalisation of that given by Umemura and Matsushima who treated the case when F has non-degenerate Chern class.

P. Scott :

The Torns Theorem

A crucial role in Thurston's work on hyperbolic structures on 3-manifolds is played by the

Torus Splitting Theorem (Johannson; Jaco-Shalen). If M^3 is compact, orientable, irreducible and sufficiently large, then there is a finite family F of adjoint incompressible tori in M such that each component of M cut along F is atoroidal or a Seifert fibre space.

This theorem follows from more general work of Johnson and Jaco and Shalen. This work is very complicated. But one can prove the splitting theorem differently by first proving the

Torus theorem. If M^3 is compact, orientable, irreducible, sufficiently large and not atoroidal, then either M admits an embedded torus not parallel to ∂M or M is a Seifert fibre space.

This result was first announced by Waldhausen and has been proved by Fenstel. It also follows from the general work of Johannson and Jaco and Shalen. A different proof of the Torus theorem was discussed.

R. Sibner :

Nonlinear Harmonic Forms, Minimal Surfaces and Negative Curvature

On an oriented Riemannian manifold one can consider "ρ-harmonic" p-forms ω which satisfy $d\omega = 0$ and $\delta\rho\omega = 0$. If $q(\omega)$ denoted the (pointwise) norm of ω, the density function $\rho = \rho(q)$ satisfies: $\frac{d}{dq}(\rho q) > 0$ for $q < q_s$ and tneding to zero as $q \nearrow q_s$. Two examples for ρ are $\rho = (1 - \frac{q-1}{2})^{\frac{1}{\gamma-1}}$ for which $q_s = \frac{2}{\gamma+1}$ or $\rho = (1+q^2)^{-\frac{1}{2}}$ for which $q_s = \infty$. It is natural to ask for differential geometry restrictions on the location of points where q has a relative maximum. If $p = 1$ and $q < q_s$ (subsonic flow) then the answer turns out to be the same as for harmonic forms ($\rho \equiv 1$). The maximum cannot be attained at a point, where the Ricci curvature is positive in the direction of the flow. A similar result holds in the case $p > 1$.

For $p = 1$, M a domain in R^2 and $\rho = (1+q^2)^{-\frac{1}{2}}$ the equation reduces to the equation for minimal surfaces in non-parametric form. A "nonlinear Hodge theorem" was stated which gives some new results in this direction.

T. Sunada :

Harmonic Mappings into a Nonpositively Curved Symmetric Space

Suppose M is a compact Kähler manifold and $\Gamma\backslash D$ is a compact quotient of a bounded symmetric domain. A holomorphic map φ of M into $\Gamma\backslash D$ is called strongly rigid if the

following holds: if Ψ is holomorphic and homotopic to φ, then $\varphi = \Psi$.

Theorem 1. If φ is as above, then φ is strongly rigid if the centralizer of the image under φ of $\pi_1(M)$ in $\pi_1(\Gamma\backslash D)$ is trivial.

Theorem 1 is an immediate consequence of

Theorem 2. Let M be a compact riemannian manifold and N a compact locally symmetric space of non-positive curvature. Denote by Harm(M,N) the set of harmonic mappings.

i) Harm(M,N) is a finite dimensional manifold; each component is compact

ii) if $x \in M$, and if C is a component of Harm(M,N), then $C \longrightarrow N$, $\varphi \longmapsto \varphi(x)$ is a totally geodesic immersion. In particular, C is locally symmetric of non-positive curvature

iii) $\pi_1(C,\varphi) \cong$ centralizer of image of $\pi_1(M)$ in $\pi_2(N)$ under φ.

S. Wolpert :

The Length Spectrum as Moduli for Compact Riemann Surfaces

Let S be a compact Riemann surface endowed with the hyperbolic metric. Denote by Evsp(S) the spectrum of the Laplacian of S and by Lsp(S) the list of lengths of closed geodesics. Trsp(S) is the list of traces of representatives

of conjugacy classes of $\pi_1(S)$. Via Selberg trace formula knowledge of Evsp(S) is equivalent to that of Lsp(S). The following question was considered: does Lsp(S) determine S?

Theorem. S is uniquely determined by Lsp(S) if $S \in T_g - V_g$, where T_g is Teichmüller space and V_g a real analytic proper subvariety of T_g.

Corollary. If Q(Trsp(S)) has transcendence degree 6g - 6 over Q, then $S \in T_g - V_g$.

A theorem of M.F. Vignèras states, that $V_g \neq \emptyset$ for some g. The proof of the above theorem requires the theory of amalgamation of Fuchsian groups.

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