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Mathematical Aspects of Computerized Tomography

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Chairmen: G. T. Herman (Buffalo)
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The image reconstruction from projection has independently arisen in a large number of scientific fields. The problem of finding the distribution of radionuclides indicating the physiological functioning of the human body, the dynamic behaviour of the beating heart of a patient,the internal structure of the solar corona, the defect of material in nondestructive testing have in common the same mathematical foundation, the Radon transform. The aim of the conference was to discuss the state of the art in the relevant parts of pure and applied mathematics and to show on the one hand to the engeneers the progress in the theory and on the other hand to the mathematicians the actual problems in the applications. The meeting started with a review on tomography and related topics of Nobel laureate for medicine 1979, Prof. A. M. Cormack. The further lectures were: mathematical analysis of the Radon transform, reconstruction from limited data, regularization and optimization methods for ill-posed problems and applications in a variety of fields.

It would have been impossible to bring together this group of specialists without the financial support of the Mathematisches Forschungsinstitut. The excellent facilities created a stimulating atmosphere which was appreciated by all the participants.

Abstracts

T. BETH:

A Finite Version of the Radon Transform - Based on Finite Geometries and Error-Correcting Codes

Discrete Radon-Transform measurements can be considered as vectors in the image of the linear mapping induced by the incidence matrices of suitable finite affine planes. The theory of error-correcting codes provides methods to "invert" this mapping by the maximum-likelihood-decoding procedure. Results on geometric codes prove that the (non-invertible) check-positions in the case of CT-pictures can be chosen to represent the known attenuation coefficient of surrounding material (e. g. water) and thus be neglected. Fast decoding algorithms can be provided.

Å. BJÖRCK:

Computational complexity of first kind integral equations in two dimensions

For the numerical solution of Fredholm integral equations of the first kind some regularization technique must be used. Suppose that discretization gives the system of linear equations $Kf = g$, where K is an $N \times N$ matrix. Then we can either use a Tikhonov regularization and solve the system $(K^T K+p^2 I)f = K^T g$ by some direct method. Alternatively an iterative regularization e. g. the Landweber iteration $f_{k+1} = f_k+\alpha_k{}^T(g-Kf_k)$ can be used.

In two dimensions the direct method is in general computationally feasible only for a rather coarse grid, say $n = N^{1/2} < 30$. However, in many important applications (including CT) the kernel of the integral equation has properties, which can be used to reduce the computational complexity (both number of arithmetic operations and storage) for the direct and iterative approach. We give complexity results for the cases when the kernel is product separable and/or of convolution type. In the latter case the matrix K will be a block Toeplitz matrix where the blocks also have Toeplitz structure. We mention a recent algorithm by which such systems can be solved in O(NlogNlogN) arithmetic operations.

Y. CENSOR:

Intervals in Linear and Nonlinear Problems of Image Reconstruction

In the "series expansion" approach to image reconstruction the mathematical formulation takes the form of a system of equations $f_i(\underset{\sim}{x}) = p_i$, $i = 1,2,...,m$, where $f_i : \mathbb{R}^n \to \mathbb{R}$, $\underset{\sim}{x} \in \mathbb{R}^n$. The Jacobian matrix of this system is generally huge, sparse, and lacks structure in its sparsity pattern. The inevitable inconsistency of the system and the fact that it describes the real problem only approximately suggest its replacement by a system of intervals $p_1 - \varepsilon_i \leq f_i(\underset{\sim}{x}) \leq p_i + \varepsilon_i$, $i = 1,2,...,m$. An approximate solution to the image reconstruction problem is found by either aiming at an arbitrary point that solves the intervals system (feasibility problem) or by solving an optimization problem over the interval constraints. We describe a method for the nonlinear feasibility problem that arises in Emission Computerized Tomography and another family of iterative algorithms for interval convex programming for the case of linear interval inequalities that arise from Transmission CT.

A. M. CORMACK:

Early Tomography and Related Topics

A personal account of the author's connection with CT in two phases. The first started in the Groote Schuur Hospital, Cape Town in 1956 and continued in a desultory fashion until the publication of two papers in '63 and '64 which attracted almost no attention. The second phase started around 1971 with the "discovery" of the work of Radon, Cramér and Wold, Bracewell and others and, of course, Hounsfield and the EMI-scanner. Developments using agencies other than X-rays (e.g. ultrasound and nuclear magnetic resonance) were discussed briefly. The use of protons in CT was discussed in more detail and comparisons of X-ray and proton scans of organs were presented.

A. M. CORMACK AND E. T. QUINTO:

A Radon Transform on Spheres through the Origin in R^n

On $C(R^n)$ we invert the Radon transform that maps a function to its mean values on spheres containing the origin. Our inversion formula implies that if $f \in C(R^n)$, and its transform is zero on spheres inside a disc centered at O, then f ist zero inside that disc. We give functions $f \in C\ (R^n)$ whose transforms are identically zero and we give a necessary condition for a function to be the transform of a rapidly decreasing function. We show that every entire function is the transform of a real analytic function. These results can be applied to solutions to the Darboux differential equation.

A. R. DAVIES:

Mathematical aspects of electron-density reconstruction in crystallography from X-ray diffraction data

The electron-density function $p(\underset{\sim}{x})$, $\underset{\sim}{x} \in V \subset \mathbb{R}^3$, within an unit cell of a crystal is defined by a Fourier series whose coefficients $F(\underset{\sim}{h})$, $\underset{\sim}{h} \in \mathbb{Z}^3$, are in general complex. In X-ray diffraction experiments only the moduli $|F(\underset{\sim}{h})|$ can be measured, while the phases are lost; same phases can usually be estimated on a sub-set $\mathbb{P} \subset \mathbb{Z}^3$ by direct (computational) or other methods. In many biophysical situations, however, $\mathbb{P}$ is small and these directly determined phases when included in the series for $p(\underset{\sim}{x})$ yield only low-resolution maps. This paper describes a self-convolution method on the Fourier lattice $\mathbb{Z}^3$ which can be used to extend phase information into $\mathbb{Z}^3 \setminus \mathbb{P}$, and hence to improve resolution. Certain features of this method should prove of direct interest to workers in CT. These include (i) imposing real-space non-negativity constraints while working in Fourier space; (ii) an iterative method for minimizing very large non linear least-squares functionals by an alternating plane block-diagonal approximation to the full Gauß-Newton normal equations; (iii) using a 3-parameter continuation method to facilitate convergence of nonlinear least-squares fitting.

U. ECKHARDT:

Semi-infinite systems of inequalities in CT

A iterative method was presented for solving semi-infinite systems of linear inequalities. This method has the properties of converging linearly and using only column information in each iteration step. Moreover, it gives an indication whether the problem under consideration has a solution or not. Numerical experiments showed that its convergence behaviour is similar to the method of Agmon so it seems well suited for applications in computerized tomography.

T. ELFVING:

Methods for entropy maximization with application to image reconstruction

In image reconstruction the relation between the object, $f(x)$, and the measured projection data, b, is given by $b = Rf$, where R is the Radon transform. It is known that a finite number of projections do not uniquely determine the object. To select a unique solution one can adopt some optimization criterion: $\max h(f)$ such that $b = Rf$. In the discrete case the object f is sampled over a rectangular grid and the operator R is a matrix, typically both large and sparse. The entropy is, for the discrate case, defined as $\Sigma f_i \ln f_i$, and can, under certain assumptions, be interpreted as the most likely solution. We will in this talk survey methods for computing the maximum entropy solution of $Rf = b$, with special attention on their efficiency for large problems.

H. W. ENGL:

Behaviour of solutions of linear equations under perturbations of the operator which do not preserve the rank

Let X be a Banach space, L_h a parameterized family of compact perturbations of the identity, $f_h \varepsilon R(L_h)$. We assume that for sufficiently small $h \neq 0$, $\dim N(L_h) = n$, but that $\dim N(L_o)$ may be $\neq n$. The problem we consider is to give reasonable sufficient conditions which ensure the convergence of solutions of $L_h x = f_h$ to solutions of $L_o x = f_o$. We assume that $L_h \to L_o, f_h \to f_o$ as $h \to 0$ and that the Riesz index of L_o equals one; let P be the projector onto $N(L_o)$ parallel to $R(L_o)$. Then, if there exist an operator $R: N(L_o) \to N(L_o)$ with $\dim N(R) \leq n$ and functions $v_1, \ldots, v_4$ such that $\|L_h - L_o\| \leq v_1(h) + v_2(h)$, $\|PL_h|_{N(L_o)} - v_1(h)R\| \leq v_3(h)$, $\|Pf_h\| \leq v_4(h)$ with $v_2/v_1 \to 0$, $v_3/v_1 \to 0$, while v_4/v_1 remains bounded as $h \to 0$, then

there exists an n-dimensional linear submanifold of the solution set of $L_o x = f_o$ all elements of which are limits of solutions of $L_h x = f_h$ under $h \to 0$.
We indicate how this result could be applied to sensitivity analysis of underdetermined systems of nearly linearly dependent linear equations as they might arise in image reconstruction problems.

R. GORENFLO:

On the continuity of the constrained pseudo-inverse

Let U be an E-space*), $Z = \mathbb{R}^m$ (with $m \in \mathbb{N}$) strictly normed, $A \in L(U,Z)$, $S \subset U$ closed and convex. Then $T := A(S)$ is convex, and S bounded => T closed. For T being closed let P_T be the metric projector $Z \to T$. For $z \in T$ the norm-minimal $\tilde{u} \in S$ with $A\tilde{u} = z$ does exist uniquely, we write $\tilde{u} =: A_S^+ z$. For T closed we set $A_S^+ a := A_S^+ P_T a$ for all $a \in Z$. We call A_S^+ the "S-constrained pseudo-inverse".
Main results: T is a polyhedron => $A_S^+ : Z \to S$ is continuous. Else A_S^+ restricted to the intrinsic core of T is continuous.
Special attention is paid to the case of U being a Hilbert space.
Applications to constrained interpolation and fitting problems and to numerical treatment of first kind integral equations are discussed.
Acknowledgment: The results have been obtained in cooperation with Martin Hilpert (Freie Universität Berlin).

*) See R. B. Holmes: A course on optimization and best approximation Springer-Verlag Berlin etc. 1972: A real Banach space U is an E-space if it is reflexive, strictly normed, and $x_n, x \in U$, $x_n \rightharpoonup x$, $\|x_n\| \to \|x\|$ => $x_n \to x$.

F. A. GRÜNBAUM:

Reconstruction with arbitrary directions: dimensions two and three

The convolution algorithm introduced in [I] is discussed with special emphasis on the relation between the angular range of the projections and the "quality" of the reconstruction.

The same method can be extended to the three dimensional case for reconstruction of a function from the integrals on all the planes normal to an arbitrary set of directions in space.

The theory developed in [I] can be properly modified to give a bound on the reconstruction error for an arbitrary choice of directions in space. Different choices of directions are considered from this point of view.
The three dimensional problem is relevant in Nuclear Magnetic Resonance.

[I] M. E. Davison and F. A. Grünbaum Convolution algorithms for arbitrary projection angles. IEEE TNS NS-26 April 1979 2670-2673

J. HEJTMANEK:

Reconstruction of Density Functions from Radiographs as an Inverse Problem in the Scattering Theory of the Linear Transport Operator

Scattering theory of the linear transport operator was initiated by J. Hejtmanek (1975) and further developed by B. Simon (1975) and J. Voigt (1976). A survey about this theory can be found in the book: M. Reed, B. Simon, Modern Methods in Mathematical Physics, vol. 3 (Scattering theory), 1979. The linear transport equation, which describes the time behavior of the photon density function in tissues for the CT model, is a simple version of the neutron transport equation, which was the focus of much mathematical work during the last 30 years for reactor engineers and neutron physicists. It is proved that the Heisenberg operator is a multiplication operator, and that it is a one-to-one mapping from the positive cone $L^1_+(\mathbb{R}^2 x S^1)$ onto itself. The inverse problem can be solved by the inverse Radon transformation formula.

G. T. HERMAN:

Surfaces of Organs in Discrete Three-Dimensional Space

Computed Tomography provides us with values (CT numbers) assigned to volume elements (voxels) which are abutting parallele-pipeds

filling a portion of three-dimensional space. Organs can be distinguished from their immediate surrounding if the CT numbers of voxels just inside the organ are different from those of adjacent voxels just outside the organ. The boundary between the organ and its surrounding is then reprensentable by a set of faces separating pairs of voxels.
In this paper we give a set of definitions for an appropriate three-dimensional discrete topology and prove some basic results. One of these yields a powerful representation of organ surfaces by a directed graph of special properties, which in its turn allows us to detect organ surfaces in multilayered computed tomograms in a computationally efficient way.

A. HERTLE:

On the problem of well-posedness of inverting the Radon Transform

We consider the Radon transform (R.t.) as an operator R from $L^1(\mathbb{R}^n)$ to $L^1(S^{n-1} \times \mathbb{R})$ and start by the assertion that the problem of inverting the R.t. cannot be well-posed on L^1: There exists a sequence (f_k) in L^1, all f_k being radial and having support in the unit ball, and (Rf_k) converges to 0 uniformly on $S^{n-1} \times \mathbb{R}$, but (f_k) converges at no point and not even weakly in L^1. Under this aspect we now study the problem in two directions

1) What further conditions must be imposed on the convergence of the Radon transforms (Rf_k) to enforce uniform convergence on the back transform space (f_k)?
2) How can the R.t. be extended (from functions to measures), such that its inverse becomes continuous under a natural topology (on measures)?

Concerning question 1) we show:

R^{-1} from $R(W^{n-1}_{\infty,0})$ to L^{∞}_{0} is continuous, i. e. uniform convergence of (Rf_k) in (n-1) derivatives implies uniform convergence of (f_k).

Concerning question 2), we extend (injectively) via the identity

$$Rf(x,p) = \frac{\partial}{\partial p} \int_{x \cdot y < p} f(y)\,dy$$

the R.t. of L^1-functions to finite measures $M(\mathbb{R}^n)$. Now the R.t. is a continuous operator R from $M(\mathbb{R}^n)$ to $M(S^{n-1} \times \mathbb{R})$. We proove:

The inverse R.t. of measures $R^{-1}: R(M(\mathbb{R}^n)) \to M(\mathbb{R}^n)$ is continuous under the vague topology on measures.

Finally, we discuss some consequences on the range of the R.t. First the range of the R.t. of L^1-functions ist not closed. Next, we note a L^1-phenomenon of the R.t.: Singular measures, as $\delta(r-1)$, are mapped on L^1-functions (or even on C^k-functions). So the back transform of a "nice" function may be highly singular. At last a characterization of the product measures on $S^{n-1} \times \mathbb{R}$ contained in the range of R shows that there are not "too many" measures in the range. Thus, in spite of (f_k) may converge (vaguely) to a measure when (Rf_k) converges uniformly, this would not occur "very often".

A. K. LOUIS:

Approximation of the Radon transform from Samples in Limited Range

In order to reduce scanning time modern X-ray scanners provide projections only in a restricted range $[0,\phi]$ with $\phi < \pi$. We consider the reconstruction of densities from p+1 complete projections in $[0,\phi]$. An extrapolation procedure is given to achieve approximations g_p of the data in the missing range in order to apply the fast reconstruction algorithms. The number of operations of this procedure is a polynomial of total degree three in the number of the data. It is shown that L_2-error of the approximated picture is of order $p^{-\alpha}$ if the original belongs to the Sobolev space H_0^α. The validity of the error estimate is investigated by numerical results. 3D-pictures of reconstructions are given and artefacts are analyzed by discussing the null space of the Radon transform for finitely many angles.

P. LUX:

Redundancy in 360° direct fan beam reconstruction

Redundancy in 360° direct fan beam reconstruction gives the possibility of reconstructing images with less information. For parallel ray geometries 180° reconstructions are standard techniques.
The problems, arising with fan beam algorithm, using only a part of the information, either in the measured projections or in angular positions, are discussed.
The paper will include background presentation of theoretical results,

main steps of mathematical derivation and demonstration of results using the algorithm for the reconstruction of mathematical phantoms and measured information.

R. B. MARR:

On two approaches to 3D reconstruction in NMR zeugmatography

In nuclear magnetic resonance (n.m.r.) zeugmatography, the primary data pertains to integrals of the unknown nuclear spin density f(X,Y,Z) over planes instead of lines in R^3. Two "natural" approaches to reconstructing f from such data are: (1) By numerical implementation of the inverse Radon transform in three dimensions (the direct approach), and (2) by application, in two successive stages, of existing well-known algorithms for inverting the two-dimensional Radon transform (the two-stage approach). These two approaches are discussed and compared, both from a theoretical startpoint and through computer results obtained with real as well as simulated n.m.r. data. For the cases studied to date the two methods appear to produce qualitatively similar results.

A. NAPARSTEK:

Continuous and discret image reconstruction formulas for fan-beam data with minimality conditions

In this paper we present mathematical results on image reconstruction from fan-beam projections in which the data set is either minimal or "nearly minimal". We present the analogue for divergent beams of the Radon Inversion formula for parallel beams over 180^o and discuss its derivation. We also give a discrete implementation of this inversion integral and demonstrate its practical limitations. We then show how the previous approach can be modified to obtain new inversion integrals, and from these suitable numerical implementations which do not have the practical limitations referred to earlier.

M. Z. NASHED:

Continuous Analogues of Iterative Methods of Cimmino and Kaczmarz for Integral Equations of the First Kind and Their Moment Discretizations

Kaczmarz's method and variants thereof have been used effectively in the numerical resolution of linear algebraic equations arising from tomography and other areas of reconstruction from projections. The method is applied to the system of equations arising from full discretization. In contrast, Cimmino's method which has universal convergence properties similar (in theory) to Kaczmarz's method, is not as widely advocated in practice. In this talk, we will present continuous analogues of the iterative methods of Cimmino and Kaczmarz for integral equations of the first kind and the integral equation of image reconstruction, subject to appropriate constraints. These analogues are studied in particular in the framework of moment discretization (rather than full discretization) of the integral equations. Some preliminary results on convergence properties will be given.

F. NATTERER:

The Attenuated Radon Transform

In (single photon) emission tomography one has to solve the integral equation

$$(R_\mu f)(s,\omega) = \int_{-\infty}^{+\infty} f(x)e^{-(M\mu)(x,\omega)}\,dt = g(s,\omega)$$

for both the activity distribution $f(x)$ and the attenuation distribution $\mu(x)$. Here, $x = s\omega + t\omega^\perp$, $\omega = (\cos\varphi, \sin\varphi)$, $\omega^\perp = (-\sin\varphi, \cos\varphi)$.

It is shown that $M\mu$ can be determined to some extent from the consistency conditions in the range of R_μ. More precisely, $(M\mu)(x,\omega)$ is determined up to an additive constant on $\mathrm{supp}(f) \times S^1$ for f a finite linear combination of Dirac measures.

M. OPPEL:

Generalized Radon Transformation

Besides Radon's paper (1917) there is a theorem of Cramer-Wold (1936) which is connected to computerized tomography. Starting from this and results of Réugi and Gilbert (1952) on the number of necessary projections the problem of reconstruction of a measure from its projection measures is discussed more generally. Under appropriate conditions the following assertion holds: If ν_n and μ are Borel measures on X, ϕ is a set of mappings $\varphi: X \to Y$ and μ is determined uniquely by $(\mu o \varphi^{-1} : \varphi \in \phi)$ then ν_n converges weakly to μ iff $\nu_n o \varphi^{-1}$ converges weakly to $\mu o \varphi^{-1}$ for every $\varphi \in \phi$. In more special cases those sets ϕ are described and procedures for reconstruction of μ from $(\mu o \varphi^{-1} : \varphi \in \phi)$ are indicated.

L. R. OUDIN:

The Radon Transform in R^2. The distributions used for elemination of an additive noise

The Radon Transform R(f) of a continuous function f with compact support is reminded. Next, the transform R(T) of a temperate distribution is expressed. Choosing distributions of rapid descent, a generalization of the properties found for classical Radon images of square integrable functions with compact support is evidenced. Namely two distributions are used in order to find the analytical correspondence between the circular harmonics of a function f and their respective images by R. The use of distributions gives rise to four classes of applications:

1) Restitution of circular harmonics from Radon image; its advantages.
2) Compatibility conditions upon circular harmonics
3) Algorithmic processes for elimination of an additive noise
4) Convolution between distributions. Applications.

At last, a Table of systematic Radon Transforms is built.

T. M. PETERS:

Resolution improvement to C.T. systems using aperture function correction

Most C.T. systems, in order to maximize photon capture utilize detectors whose size tends to limit the attainable resolution of the system. This paper demonstrates the relationships between the aperture function, sampling and the resolution in the reconstructed image, and shows that by judicious application of Wiener filtering techniques, significant improvement in resolution may be attained. Practical demonstrations using a positron emission tomography system are presented.

E.-P. RÜHRNSCHOPF:

Nonlinearity and Inhomogenity Effects due to the Exponential Attenuation of Radiation

Ein CT-Meßsystem liefert nur eine 'exponentiell verschmierte Radon-Transformierte' der gesuchten Funktion μ:

(1) $\tilde{R}\mu = -\log \int e^{-R\mu(\cdot,\eta)}\, dW(\eta)$.

In dieser Form lassen sich verschiedene Effekte beschreiben, wenn man die Verteilung W z. B. als Detektorempfindlichkeitsprofil, als Energiespektrum oder als statistische Verteilung von Mikrostrukturen interpretiert. Aus (1) folgt die fundamentale Ungleichung

(2) $\tilde{R}\mu \le \bar{R}\mu = \int R\mu\, dW$.

Abschätzungen für den Nichtlinearitätsfehler, Korrekturverfahren etwa für die Strahlaufhärtung sowie Bildbeispiele über typische Phänomene bei der Bildrekonstruktion werden gegeben.

Über die Nichtlinearität hinaus führt die Berücksichtigung der Zufallsnatur der Strahlungsschwächung und Meßwertgewinnung zu einem inhomogenen stochastischen Prozeß mit typischen Auswirkungen auf die Rauschtextur des rekonstruierten Bildes.

H. SCHOMBERG:

Solved and unsolved problems in nonlinear object reconstruction from projections

Scanning a patient's cross-section with an X-ray CT-machine is mathematically modelled by the Radon transform, and reconstructing it amounts to numerically inverting the Radon transform. If, however, projections are taken by means of ultrasound waves or electric currents (instead of X-rays), then the appropriate mathematical models become certain nonlinear variations of the Radon transform, or, alternatively, "direct" problems for certain partial differential equations. The reconstruction then amounts to finding an acceptable solution of the respective, nonlinear "inverse" problems. Some results (positive as well as negative) on the uniqueness of the reconstructions in case of ideal data are reported. Also, numerical methods for practically computing a solution are presented, along with numerical test results. Due to Fermat's principle the situation in ultrasound CT is still manageable, whereas in electrical CT pessimism is appropriate.

G. SCHWIERZ:

Sampling - and Discretization - Problems in X-ray CT

Sampling- and discretization - problems are of great significance for the image reconstruction by means of digital computers.

By the aid of a fundamental theorem of communication theory, the so called sampling theorem, the question of how fine the discretization should be chosen, can be answered. It is shown, that due to the finite number of measurements the filtering process caused by the limited width and heigth of the detectorelements is a necessity for the reconstruction of a clear and undisturbed image. Furthermore, if the measurementsystem fullfills the Nyquist condition, it can be shown, that by suitable choice of the discrete kernel and the adequate interpolation of filtered data before backprojection, the discrete convolution performs exactly the filtering process, which is demanded by theory.

D. C. SOLMON:

Stability and consistency Conditions for the divergent Beam and parallel Beam X-ray Transforms

Let $P_\theta f(x) = \int_{-\infty}^{\infty} f(x+t\theta)dt$, $D_a f(\theta) = \int_0^{\infty} f(a+t\theta)dt$, $\theta\in S^{n-1}, x\in\theta^\perp, a\in R^n$, be the parallel beam and divergent beam X-ray transforms respectively. For a finite set of directions $(\theta_1,\ldots,\theta_m)$ or sources $(a_1,\ldots,a_m)$, let $P = (P_{\theta_1},\ldots, P_{\theta_m})$ and let $D = (D_{a_1},\ldots D_{a_m})$. The operators P and D are studied as maps between appropriate Hilbert spaces. Sufficient conditions that P and D have closed range (and hence a continuous generalized inverse) are given. Also, consistency conditions are given for the range of these operators.

W. SWINDELL:

An Analog Implementation of the Inverse Radon Transform

I have described an (incoherent) optical analog computer that performs the filtered back projection reconstruction method to clinically derived X-ray transmission data. Data are recorded on photo graphic film in sinogramformat with high spatial resolution. The film also serves to take the logorithm of the X-ray intensity as required by the algorithm. The hardware uses optical convolution methods to effect the algorithm and the output image is displayed on a CRT in special display format. Our system has a high detective quantum efficiency of about 70 % and we now obtain image reconstruction that are equivalent in-spatial, density resolution close to those obtained with commercial digital scanners like the digital systems. We are limited only by photon countery statistics.

O. J. TRETIAK:

A Model for Optimal Reconstruction

The problem of inverting the Radon transform from discrete data is considered as the approximate evaluation of a linear functional whose domain is a bounded subset of a linear space. A penalty function is defined which evaluates both random and systematic errors, and the optimal reconstructional is obtained as the argument of a min-max problem. Several simplifications are developed for this extremal problem, and the method is applied to find optimal reconstruction kernels for some problems with noise and/or missing data.

W. WAGNER:

Reconstructions from incomplete scan data

The quality of CT-pictures is limited by the applied patient dose. To reduce dose, the collection of scan data should be restricted to that region which is of medical interest (partial scan data). However, because data belonging to ray paths outside the region are then missing, usual reconstruction algorithms produce severe picture distoritions. Recently, a method has been introduced which compensates for this (e. g. W. Wagner, IEEE Trans. Nucl. Sc., NS 26, 79, 2866 and R. M. Lewitt, Med. Phys. Vol. 6, 79, 412). Its basic idea is to combine artificial data, which satisfy some physical constraints and fit the slice outline, with the partial data before starting the reconstruction. An improved version of this method is now described. Images which were reconstructed from real patient scans (skull, lung and abdomen), including one scan of outstandingly high spatial resolution, are presented.

G. WAHBA:

Regularization, Cross Calidation and the Landweber Iteration For Large Linear Systems

Consider the linear system $z_{nx1} = K_{nxp}f_{px1}+e_{nx1}$, where the subscripts indicate dimensions. Let S_{nxn} be a rank n matrix and Q_{pxp} be strictly positive definite. A regularized estimate f^{λ} of f is the minimizer of $||S(z-Kf)||^2 + \lambda f'Q^{-1}f$ and is given by

$$f^{\lambda} = Q^{1/2}(\tilde{K}'\tilde{K}+\lambda I)^{-1}\tilde{K}'\tilde{z} = \sum_{j=1}^{\ell} (1+\lambda/d_j^2)^{-1}\frac{(Sz,u_j)}{d_j}Q^{1/2}v_j$$

where $\tilde{K} = SKQ^{1/2}, \tilde{z} = Sz$, and the u_j, v_j, are the left and right eigenvectors of K corresponding to the ℓ non zero eigenvalues $d_1,\dots,d_{\ell}$. Assuming that $Ee = 0$, $EeS'Se = \sigma^2 I$, then the generalized cross validation (GCV) estimate of λ is the minimizer of $V(\lambda)$ given by

$$V(\lambda) = \frac{\frac{1}{n}||\tilde{z}-\tilde{K}\tilde{f}^{\lambda}||^2}{(\frac{1}{n}Tr(I-\tilde{K}(\tilde{K}'\tilde{K}+\lambda I)^{-1}\tilde{K}')^2} = \frac{\frac{1}{n}||Sz-Skf^{\lambda}||^2}{(1-\frac{1}{n}\sum_{j=1}^{\ell}\frac{1}{1+\lambda/d_j^2})}, \tilde{f}^{\lambda} = Q^{-1/2}f^{\lambda}.$$

The calculation of the GCV estimate of λ is out of the question for large n,p. Consider the generalized Landweber iteration

$$f^k = f^{k-1} + \beta QK'S'(Sz-SKf^{k-1}).$$

Letting $\tilde{f}^k = Q^{-1/2}f^k$, this can be rewritten

$$\tilde{f}^k = (I-\beta\tilde{K}'\tilde{K})\tilde{f}^{k-1} + \beta\tilde{K}'\tilde{z}$$

and it is known that

$$\tilde{f}^k = \sum_{j=1}^{\ell} (1-(1-\beta d_j^2)^{k-1})\frac{(\tilde{z},u_j)}{d_j} v_j$$

provided $\beta < 2/d_1^2$, hence

$$f^k = \sum_{j=1}^{\ell} (1-(1-\beta d_j^2)^{k-1})\frac{(Sz,u_j)}{d_j} Q^{1/2}v_j.$$

Thus the damping factor $(1+\lambda/d_j^2)^{-1}$ in the regularized estimate is replaced by the damping factor $(1-(1-\beta d_j^2)^{k-1})$ in the generalized Landweber k^{th} iterate. The pair (k,β) play the role of the regularization parameter λ in damping out "high frequencies". The new result is that the GCV estimate of (k,β), which shares the optimality properties of GCV estimates, is the minimizer of

$$V(k,\beta) = \frac{\frac{1}{n}\|\tilde{z}-\tilde{K}f^k\|^2}{(1-\frac{1}{n}\sum_{j=1}^{\ell}(1-(1-\beta d_j^2)^{k-1}))^2} = \frac{\frac{1}{n}\|Sz-SKf^k\|^2}{[(1-\frac{1}{n}Tr(I-(I-\beta SKQK'S')^{k-1})]^2}.$$

A search for the minimizing k for several choices of β is thus feasible for fairly n and p. Similar results apply to the compressed Landweber iteration.

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