MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 19/1981

General Inequalities

April 26 - May 2, 1981

The Third International Conference on General Inequalities was held from April 26 to May 2 at the Mathematisches Forschungsinstitut Oberwolfach. Only two of the three chairmen of the conference, Prof. E.F.Beckenbach (Los Angeles), Prof. M.Kuczma (Katowice) and Prof. W.Walter (Karlsruhe), were able to attend. Prof. Kuczma is seriously ill and his usual stimulating scientific contributions were badly missed. As usual, Prof. R.Ger (Katowice) served extremely well as a secretary of the Conference.

The meeting was attended by 39 participants from 12 countries. It was opened by E.F.Beckenbach, who presented the good news that the next conference on General Inequalities has tentatively been set here in Oberwolfach for the week of May 8-14, 1983.

Many branches of mathematics and its applications were represented, such as functional and differential inequalities, convexity and its generalizations, inequalities in functional analysis, in particular in sequence spaces, applications to geometry, complex variables, probability theory and economics. Classical inequalities continued to be a steady source of the new ideas and methods. Special emphasis was placed on majorization and optimization techniques which play such an important role in economic and industrial applications.

The exchange of ideas was especially **fruitful** in the Problems and Remarks sessions.

The view was shared by all participants that the most harmonic and stimulating atmosphere prevailed, resulting in many fruitful scientific discussions.

In closing remarks W.Walter expressed the gratitude of all participants for the excellent working conditions in the Institute and for the hospitality of its leaders and staff.

E.F.Beckenbach

W.Walter

The abstracts of the talks, the remarks and the problems follow (separately) in chronological order of presentation.

R. REDHEFFER: <u>Easy proofs of hard inequalities</u>

A few inequalities, of general interest, are proved by methods which are perhaps shorter than those sometimes used. The talk is, in part, expository.

R. J. NESSEL: <u>On uniform boundedness principles with rates</u>

Let X be a Banach space, Y a normed linear space, and $U \subset X$ a linear subspace with seminorm $|\ |_U$. Consider the intermediate spaces $U \subset X_\omega \subset X$, $X_\omega := \{ f \in X;\ K(t,f;X,U) = O(\omega(t))\ ,\ t \to 0+ \}$, where the K-functional is defined for $f \in X$, $t \geqslant 0$ by

$$K(t,f;X,U) := \inf \{ \| f - g \|_X + t\, |g|_U \ ;\ g \in U \} \ ,$$

and ω is a modulus of continuity satisfying $t/\omega(t) = o(1)$, $t \to 0+$. Let $\lim_{n \to \infty} \varphi_n = 0$ monotonely.

<u>Theorem</u>: For $X, Y, U, X_\omega, \omega, \varphi$ as above let $\{R_n\} \subset [X,Y]$ be a sequence of bounded linear operators of X into Y and suppose that there exists $\{h_n\} \subset U$ such that

$$\| h_n \|_X \leqslant c_1 \ , \qquad | h_n |_U \leqslant c_2 \varphi_n^{-1} \ ,$$

$$\| R_n h_n \|_Y \geqslant c_3 \| R_n \|_{[X,Y]} \ .$$

If for every $f \in X_\omega$ one has $\|R_n f\|_Y = o(1)$, $n \to \infty$, then the operator norms necessarily satisfy the growth condition

$$\omega(\varphi_n) \; \|R_n\|_{[X,Y]} = o(1) \quad .$$

The method of proof essentially consists in the familiar gliding hump method, but now equipped with rates. The present approach simplifies and unifies the discussion of the sharpness of error bounds in various areas of analysis. Explicit applications are given to numerical quadrature, multipliers of uniform convergence, and to the numerical solution of properly posed initial value problems.

F. FEHÉR: Exponents of submultiplicative functions and function spaces

If L_p $(1 \leq p \leq \infty)$ is the Lebesgue space of real valued, measurable functions with the usual norm, then it is well known that the number p plays an important role in connection with properties such as reflexivity, uniform convexity and separability. On the other hand, quite a lot of generalizations of L_p spaces are known such as Lorentz spaces, Orlicz spaces, Zygmund spaces. A naturally arising question therefore is, whether it is possible to assign also to these more general spaces numbers which would correspond to the number p in the Lebesgue case. The purpose of this talk therefore is, first to give a brief survey on the different attempts, which were made in literature in order to define such numbers. In the second part of the talk it is shown, how all these numbers can be obtained from one basic principle on submultiplicative functions.

K. ZELLER: Erweiterung des Mittelwertsatzes von Riesz

Der Mittelwertsatz von M. Riesz spielt eine wichtige Rolle in verschiedenen Gebieten der Mathematik (Fractional Calculus, Limitierung, Funktionalanalysis; Bosanquet, Wilansky, Beekmann u.a.) . Leider gilt er nur für den Exponentenbereich $(-1,0]$. Hier wird gezeigt, daß der Satz sich in geeigneter Weise (Mittelung) auf den Bereich $(0,\infty)$ erweitern läßt. Ein Beweis beruht auf entsprechenden Positivitatsaussagen über Summen von Binomialkoeffizienten.

J. ACZÉL: Functional equations and inequalities in "rational group decision making" (Joint work with Pl. Kannappan, C. T. Ng and C. Wagner)

A fixed amount s is to be allocated to a fixed number m (> 2) of competing projects. A group of n decision makers, each make recommendations, allocating, say, $\underline{x}_i = (x_{i1},\dots,x_{in})$ to the i-th project, in order to establish the "consensus" allocation $f_i(\underline{x}_i)$. We only suppose that $f_i(\underline{0}) = 0$ ("consensus of rejection") and prove that each f_i is the same weighted arithmetic mean.

This result may be surprising since so little has been supposed. But the fact that the whole amount s should be allocated, gives $\sum_{i=1}^{m} \underline{x}_i = \underline{s} := (s,\dots,s) \Longrightarrow \sum_{i=1}^{m} f_i(\underline{x}_i) = s$ which can be written as a functional equation. This is still not enough, but the inequalities $0 \leqslant f_i(\underline{x}_i)\ (\leqslant s)$ do the trick ($\underline{x}_i \geqslant \underline{0}$; $i = 1,\dots,m$).

The cases $m \leqslant 2$ are also completely solved.

I. FENYÖ: An error estimate for approximate solutions of operator equations

Let $(A, \|\cdot\|_A)$ and $(B, \|\cdot\|_B)$ be normed spaces and consider the equation

(1) $$\mathcal{M}x = f \qquad (f \in B)$$

where $\mathcal{M}$ is a given linear operator. We approximate this by

(2) $$\mathcal{N}y = g \qquad (g \in B).$$

Also $\mathcal{N}$ is supposed to be linear from A to B. If we suppose

a) $\mathcal{N}^{-1}$ exists

b) $\| \mathcal{N}^{-1}(\mathcal{N} - \mathcal{M}) \| < 1$,

then (1) has a solution and for the difference of the solutions of (1) and (2) the inequality

$$\|x - y\|_A \leqslant \frac{\|\mathcal{N}^{-1}(\mathcal{N}-\mathcal{M})\| \, \|\mathcal{N}^{-1}f\|_A}{1 - \|\mathcal{N}^{-1}(\mathcal{N}-\mathcal{M})\|} + \|\mathcal{N}^{-1}(f-g)\|_A$$

holds. Special important situations are discussed.

R. GER: Almost approximately additive mappings

The functional inequality of approximate additivity was first considered by D. H. Hyers (On the stability of the linear functional equation, Proc. Nat. Acad. Sci. USA 27 (1941) , pp. 222-224). He proved that to each function f mapping a given Banach space E into an another Banach space E′ and satisfying the inequality

$$\| f(x+y) - f(x) - f(y) \| \leqslant \varepsilon \quad , \; x,y \in E,$$

there corresponds an additive function $\ell : E \longrightarrow E'$ such that

$$\| f(x) - \ell(x) \| \leqslant \varepsilon \quad , \; x \in E.$$

This kind of problem has been later investigated by many authors. During the preceding meeting on General Inequalities (Oberwolfach, 1978) J. Rätz presented some nice results of that type in a very general setting (cf. Proceedings of the Second International Conference on General Inequalities, Edited by E. F. Beckenbach, ISNM 47, pp. 233-251).

Answering a question of L. Reich we present a result about the behaviour of functions satisfying the approximate additivity inequality almost everywhere with respect to an axiomatically given family of "small" sets (almost approximately additive functions). Our considerations are carried on under assumptions similar to those adopted by J. Rätz in his paper quoted above.

Z. PÁLES: Hölder and Minkowski inequality for homogeneous means depending on two parameters

Let $a,p \in \mathbb{R}$, $\underline{x} = (x_1,\dots,x_n) \in \mathbb{R}_+^n$ and

$$M_{n,a}(\underline{x})_p = \begin{cases} \left[\sum_{i=1}^{n} x_i^{a+p} \Big/ \sum_{i=1}^{n} x_i^p \right]^{\frac{1}{a}} , & a \neq 0 , \\ \exp \left[\sum_{i=1}^{n} x_i^p \log x_i \Big/ \sum_{i=1}^{n} x_i^p \right] , & a = 0 . \end{cases}$$

We investigate the following inequalities:

1) $$M_{n,a_0}(\underline{x}_1 + \dots + \underline{x}_k)_{p_0} \leqslant M_{n,a_1}(\underline{x}_1)_{p_1} + \dots + M_{n,a_k}(\underline{x}_k)_{p_k} ,$$
$$(\underline{x}_1,\dots,\underline{x}_k \in \mathbb{R}_+^n , \; k \in \mathbb{N}) ,$$

2) $$M_{n,a_0}(\underline{x}_1 \cdot \dots \cdot \underline{x}_k)_{p_0} \leqslant M_{n,a_1}(\underline{x}_1)_{p_1} \cdot \dots \cdot M_{n,a_k}(\underline{x}_k)_{p_k} ,$$
$$(\underline{x}_1,\dots,\underline{x}_k \in \mathbb{R}_+^n , \; k \in \mathbb{N})$$

and their inverses. We give necessary and sufficient conditions concerning the parameters $(a_o,p_o), \dots, (a_k,p_k) \in \mathbb{R}^2$, $k \in \mathbb{N}$ for 1) and 2) to hold.

L. LOSONCZI: Remarks on Hölder and Minkowski inequalities

We consider the following Hölder type inequality

$$(1)\quad \frac{1}{n}\sum_{i=1}^{n} x_i y_i \leqslant M_F(x)\, M_G(y) \quad \Big(x = (x_1,\dots,x_n) \in R_+^n ,\; y = (y_1,\dots,y_n) \in R_+^n ;\; n = 1,2,\dots \Big)$$

where F,G are differentiable deviation functions on $R_+ = (0,\infty)$ and M_F, M_G are the corresponding deviation means (for the definitions see Z. Daróczy, Über eine Klasse von Mittelwerten, Publ. Math. Debrecen 19 (1972) 211-217; ——, A general inequality for means, Aequationes Math. 7 (1972) 16-21).

If (1) holds then there exist homogeneous deviation means M_{F_1}, M_{G_1} such that

$$\frac{1}{n}\sum_{i=1}^{n} x_i y_i \leqslant M_{F_1}(x)\, M_{G_1}(y) \leqslant M_F(x)\, M_G(y) \quad \Big(x,y \in R_+^n ;\; n = 1,2,\dots \Big).$$

The deviation functions F_1, G_1 have the form $F_1(u,v) = v\, f_1\!\left(\frac{u}{v}\right)$, $G_1(u,v) = v\, g_1\!\left(\frac{u}{v}\right)$, $u,v \in R_+$, where f_1, g_1 are increasing convex functions which, apart from a translation, satisfy the Young inequality.

If

$$M_F(x) = \varphi^{-1}\left(\frac{\sum_{i=1}^{n}\varphi(x_i)}{n}\right), \qquad M_G(y) = \psi^{-1}\left(\frac{\sum_{i=1}^{n}\psi(y_i)}{n}\right)$$

are quasiarithmetic means where φ, ψ are twice differentiable on R_+ and $\varphi', \psi' > 0$ then there exists a constant $p > 1$ such that

$$\frac{1}{n}\sum_{i=1}^{n} x_i y_i \leqslant \left(\frac{1}{n}\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \left(\frac{1}{n}\sum_{i=1}^{n} y_i^q\right)^{\frac{1}{q}} \leqslant M_F(x)\, M_G(y)$$

$\left(x,y \in R_+^n ;\; n = 1,2,\dots \right)$ holds $\left(\frac{1}{p} + \frac{1}{q} = 1 \right)$.

I. OLKIN: Inequalities: Some applications of majorization

Majorization is a partial ordering between vectors. Because the class of order-preserving functions is rich, many different inequalities can be generated. In this talk we provide a review of the history of majorization and give a number of examples of majorization and consequent inequalities in matrix theory, probability, numerical analysis, combinatoric, as well as analytic inequalities.

Ch. WANG: Inequalities and mathematical programming

Extensions of the R-P (abbreviation of Rado and Popoviciu) inequalities, which in turn generalize the usual A-G (abbreviation of arithmetic and geometric) inequality, are given. These extensions are established by suitable equalities. Mathematical programming problems involving an MIC (abbreviation of monotonically increasing convex) function are also examined.

M. GOLDBERG: Better stability bounds for Lax-Wendroff schemes in several space variables

Some classical inequalities are used in order to obtain improved stability regions for the well-known Lax-Wendroff finite-difference approximation to the multi-dimensional hyperbolic initial-value problem

$$\partial u(x_1,\ldots,x_d,t)/\partial t = \sum_{j=1}^{d} \partial u(x_1,\ldots,x_d,t)/\partial x_j$$

$$u(x_1,\ldots,x_d,0) = f(x_1,\ldots,x_d) .$$

This talk represents joint work with E. Tadmor.

W. EICHHORN: Tax progression and decrease of income inequality

Let $\mathbb{R}_+$ be the nonnegative reals. An income tax rate is a function $p: \mathbb{R}_+ \longrightarrow [0,1)$ that assigns to each gross income $x \in \mathbb{R}_+$ a real number $p(x)$ such that $x \cdot p(x)$ is the income tax amount. Then $(1 - p(x))x$ is the income after income tax or net income.

From the (economic) policy point of view, an income tax rate should satisfy the following properties.

(i) Progression: $p: \mathbb{R}_+ \longrightarrow [0,1)$ is monotonically increasing, $p(x) \not\equiv$ const.

(ii) Preservation of motivation: Net income is a strictly increasing function of gross income, that is, $x \longmapsto (1-p(x))x$ is strictly increasing $(p: \mathbb{R}_+ \longrightarrow [0,1))$.

(iii) Decrease of income inequality: $p: \mathbb{R}_+ \longrightarrow [0,1)$ shall be such that for all income distributions $x_1,\dots,x_n$ the inequality of the distribution of net incomes $(1-p(x_1))x_1, \dots, (1-p(x_n))x_n$ is equal or smaller than that of $x_1,\dots,x_n$.

Definition (Majorization property; see, for instance, A. W. Marshall and I. Olkin: "Inequalities: Theory of majorization and its applications", Academic Press 1979): $y = (y_1,\dots,y_n) \neq 0$ satisfying $0 \leqslant y_1 \leqslant \dots \leqslant y_n$ is less unequally distributed than $x = (x_1,\dots,x_n) \neq 0$ satisfying $0 \leqslant x_1 \leqslant \dots \leqslant x_n$ if

$$\sum_{i=1}^{k} y_i \Big/ \sum_{j=1}^{n} y_j \geqslant \sum_{i=1}^{k} x_i \Big/ \sum_{j=1}^{n} x_j \qquad \text{for every } k = 1,\dots,n .$$

Remarks. Note that neither (i) nor (ii) implies (iii), that (i) and (ii) are independent and that there exist functions $p: \mathbb{R}_+ \longrightarrow [0,1)$ which satisfy (i) and (ii).

Theorem. (i), (ii) $\Longrightarrow$ (iii).

A. CLAUSING: Quasiconvexity and integral inequalities

Cargo has pointed out the "vertex phenomenon", a feature common to the proofs of many complementary inequalities. Consider, for example, the inequality of Wilkins:

$$\Phi(f) = \int_0^1 f(x)\,dx \int_0^1 \frac{1}{f(x)}\,dx \leqslant \frac{\left(\dfrac{b\,\log(b/a)}{b-a} - \dfrac{a+b}{2b}\right)^2}{2(b-a)\left(\dfrac{\log(b/a)}{b-a} - \dfrac{1}{b}\right)} ,$$

which is true for every f in the set K of all concave functions with values in $[a,b]$ $(0 < a < b)$. Here, equality holds if f is a certain extremal point of K. Φ, however, is not a convex function. Similarly, inequalities of Schweitzer, Kantorovich, Specht, Cargo/Shisha, Beckenbach, Berwald, Marshall/Olkin and others can be considered as stating that a certain nonconvex function Φ attains its supremum over some convex and compact set K at an extremal point of K. We propose a simple, unified proof of these as well as some new inequalities that is based on two facts:

(i) The functions Φ in question are quasiconvex.

(ii) For (l.s.c.) quasiconvex functions, a boundary maximum principle holds.

Also, some extensions and limitations of this method are discussed.

M. LACZKOVICH: A generalization of Kemperman's inequality

Let $f: R \longrightarrow R$ be a real function and suppose that

(1) $$f(x) \leq \sum_{i=1}^{k} a_i f(x+ih)$$

holds for every x and for every $h > 0$, where $a_i \geq 0$ and $\sum_{i=1}^{k} a_i = 1$.

Then f is increasing. Also, if

(2) $$f(x) \leq p f(x-h) + (1-p) f(x+h)$$

holds for every x and for every $h > 0$, where $0 < p < \frac{1}{2}$, then f must be increasing. These theorems (the proofs of which use different methods: number theory and real functions, respectively) rise the following general problem: what are the linear inequalities of type

$$\sum_{i=1}^{k} a_i f(x+b_i h) \geq 0$$

implying monotonicity ?

P. VOLKMANN: Existenz einer zwischen zwei Funktionen v,w gelegenen Lösung von Funktionalgleichungen der Form $u(\Phi(x_1,\dots,x_n)) = \varphi(u(x_1),\dots,u(x_n))$, wenn v,w entsprechenden Funktionalgleichungen genügen

(Bericht über gemeinsame Arbeit mit Herbert Weigel).

Sei M eine Menge. Zwei Funktionen $\Phi : M^m \longrightarrow M$, $\Psi : M^n \longrightarrow M$ heißen vertauschbar ($\Phi\Psi = \Psi\Phi$), wenn

$$\Phi(\Psi(x_{11},\dots,x_{1n}),\dots,\Psi(x_{m1},\dots,x_{mn})) =$$
$$= \Psi(\Phi(x_{11},\dots,x_{m1}),\dots,\Phi(x_{1n},\dots,x_{mn}))$$

($x_{\mu\nu} \in M$) gilt.

Sei $\Gamma \subseteq \{(\Phi,\varphi) \mid \Phi : M^n \longrightarrow M,\ \varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$, φ stetig und in jeder Variablen (schwach) wachsend, $n = 1,2,\dots\}$ mit $\Phi\Psi = \Psi\Phi$, $\varphi\psi = \psi\varphi$ für $(\Phi,\varphi), (\Psi,\psi) \in \Gamma$. Zusätzlich gelte:

(*) Für $(\Phi,\varphi) \in \Gamma$ ist φ bezüglich jeder Variablen streng wachsend oder konstant.

Satz. Sind $v,w: M \to \mathbb{R}$ mit $v \leq w$, $v(\Phi(x_1,\dots,x_n)) \geq \varphi(v(x_1),\dots,v(x_n))$, $w(\Phi(x_1,\dots,x_n)) \leq \varphi(w(x_1),\dots,w(x_n))$ für alle $(\Phi,\varphi) \in \Gamma$, so gibt es ein $u: M \to \mathbb{R}$ mit $v \leq u \leq w$, $u(\Phi(x_1,\dots,x_n)) = \varphi(u(x_1),\dots,u(x_n))$ für alle $(\Phi,\varphi) \in \Gamma$.

Es wird noch diskutiert, wie weit man sich von der einschränkenden Bedingung (*) lösen kann.

J. RÄTZ: On Lorentz transformations in the plane

Lorentz transformations in R^n have been characterized by many authors and in several ways: 1) as isometries, 2) as mappings preserving a single non-zero distance, 3) as light cone preserving mappings.

Our contribution belongs to the third circle of ideas and exhibits a big contrast of the situation in R^2 with respect to that in R^n $(n \geq 3)$. All bijective light cone preserving mappings $T: R^2 \to R^2$ are determined, and it is shown what kind of regularity conditions are used to single out the Lorentz transformations of the plane R^2. Among these the preservance of a distance inequality plays a central role.

Z. DARÓCZY: Inequalities for deviation means

Let $I \subseteq \mathbb{R}$ be an open interval. The function $E: I^2 \to \mathbb{R}$ is said to be a deviation on I if it has the following properties:

(E1) For every fixed value $x \in I$ the function $y \mapsto E(x,y)$ is strictly decreasing and continuous on I ;

(E2) $E(x,x) = 0$ for all $x \in I$.

Denote by $\mathcal{E}(I)$ the set of all deviations on I. It is known that for every $\underline{x} = (x_1,\dots,x_n) \in I^n$ the equation

$$\sum_{i=1}^{n} E(x_i,y) = 0$$

has exactly one solution $y_o \in I$ and this solution satisfies the inequality

$$\min(\underline{x}) \leq y_o \leq \max(\underline{x}) .$$

The quantity $y_o := \mathfrak{M}_{n,E}(\underline{x})$ is said to be the deviation mean value generated by $E \in \mathcal{E}(I)$. In the lecture we shall investigate some general inequalities for deviation means.

A. W. MARSHALL: Use of majorization and Schur-convexity to obtain inequalities

Majorization can be regarded as a pre-ordering of n-dimensional real vectors. Real functions of n arguments that are increasing in the sense of majorization (i.e., order-preserving functions) are said to be Schur-convex. These functions are quite well characterized, and numerous specific examples can be given. The evaluation of a Schur-convex function at two vectors ordered by majorization yields an inequality. A number of inequalities are derived in this manner to illustrate the method.

A few generalizations of majorization are also introduced, together with the corresponding order-preserving functions. Again, some inequalities are derived by comparing the values an order-preserving function takes on at two ordered points. The purpose is to expose a method for deriving inequalities, with specific inequalities obtained only for illustration.

C. ALSINA: A functional inequality for triangle functions

We study solutions of the functional inequality

(*) $$\tau(F \circ G, H \circ K) \geq \tau(F,H) \circ \tau(G,K)$$

where F,G,H and K are arbitrary distribution functions in Δ^+, $\circ$ denotes composition and the unknown binary operation τ on Δ^+ is an ε-strict triangle function.

The main result is the following

Theorem. If an ε-strict triangle function τ satisfies (*) then there exist $T_\tau \in \mathcal{Z}$ and $L_\tau \in L_0$ such that $\tau \geq \tau_{T_\tau, L_\tau}$, i.e.,

$$\tau(F,G)(x) \geq \sup \{ T_\tau (F(u), G(v)) \mid L_\tau(u,v) = x \},$$

and T_τ, L_τ are defined, respectively, by

$$T_\tau(x,y) = \tau(A_x, A_y)(1) \quad , \quad L_\tau(a,b) = \tau(\varepsilon_a, \varepsilon_b)^\vee (1/2) .$$

Moreover, all $\tau_{T,L}$ operations are solutions of (*) .

B. SCHWEIZER: Menger betweenness in α-simple spaces

Let $(S,d,G;\alpha)$ be the α-simple space generated by the metric space (S,d) and the distribution function G, and suppose $\alpha > 1$. Then the point q is Menger-between the points p and r if and only if p,q,r are distinct and

Gefördert durch DFG Deutsche Forschungsgemeinschaft

$$(*) \qquad d^{\alpha}(p,q)\, H(u) + d^{\alpha}(q,r)\, H(v) \leqslant d^{\alpha}(p,r)\, H(u+v) \quad ,$$

for all $u,v \geqslant 0$, where $H(x) = G^{-1}\left(1 - G(x^{1-\alpha})\right)$. We study the properties of the set $B(p,r)$ of all points q satisfying $(*)$. We obtain best-possible upper and lower bounds for $B(p,r)$. We show that if $(S, \|\cdot\|)$ is a normed linear space and $d(p,q) = \|p - q\|$, then $B(p,r)$ is convex and p,r are on the boundary of $B(p,r)$, but that this need not be the case when the metric d is not derived from a norm.

R. P. AGARWAL: Some inequalities for a function having n zeros

Let $x(t) \in C^{(n)}[a_1,a_r]$, satisfying

$$x(a_i) = x'(a_i) = \ldots = x^{(k_i)}(a_i) = 0 \ , \qquad 1 \leqslant i \leqslant r \ ,$$

$$a_1 < a_2 < \ldots < a_r \ , \quad k_i \geqslant 0 \ , \quad r \geqslant 2 \ , \quad \sum_{i=1}^{r} k_i + r = n \ .$$

Then

$$|x^{(k)}(t)| \leqslant C_{n,k}(a_r - a_1)^{n-k} \max_{a_1 \leqslant t \leqslant a_r} |x^{(n)}(t)|$$

$$k = 0,1,\ldots,n-1 \ ,$$

where, if $\alpha = \min(k_1,k_r)$,

$$C_{n,k} = \frac{1}{(n-k)!}\,\frac{(n-\alpha-1)^{n-\alpha-1}(\alpha-k+1)^{\alpha-k+1}}{(n-k)^{n-k}} \ , \quad k = 0,1,\ldots,\alpha \ ,$$

$$C_{n,\alpha+k} = \frac{k}{(n-\alpha)(n-\alpha-k)!} \ , \qquad k = 1,2,\ldots,n-\alpha-1 \ .$$

The constants $C_{n,k}$ are best possible if $\alpha = 0$; $k = 0$, $\alpha \neq 0$. The case $k \neq 0$, $\alpha \neq 0$ is an open problem.

E. R. LOVE: Inequalities between norms in sequence spaces

Inequalities of the form $\|Ax\|_q \leqslant C\,\|x\|_p$, where A is a linear operator, have received much less attention when x is a sequence in ℓ^p than when x is a function, at any rate since the days of Hardy, Littlewood and Polya (HLP). Interest in sequence cases has however been reviving, for instance in the work of Johnson and Mohapatra (1980).

The best such inequality in HLP is probably Theorem 318 (c). It is however restricted to $q = p$; and it does not seem to include Hardy's Inequality as a special case. It also imposes <u>two</u> monotony conditions on the elements a_{mn} of the matrix A.

I have tried to obtain theorems with looser and simpler requirements. The main one is as follows.

Theorem. If $q > p > 1$, $r^{-1} = 1 - (p^{-1} - q^{-1})$, $\alpha(t)$ is non-negative, increasing in $[0,1)$ and decreasing in $(1,\infty)$,

$$|a_{mn}| \leqslant \frac{1}{m^{1/r}} \alpha\left(\frac{n-1}{m}\right) \text{ if } n \leqslant m , \quad |a_{mn}| \leqslant \frac{1}{(n+1)^{1/r}} \alpha\left(\frac{n}{m}\right) \text{ if } n > m$$

and

$$C = \left(\int_0^1 \left(\frac{1}{t} + 1\right)^{\frac{r}{q}} \alpha(t)^r \, dt\right)^{\frac{1}{r}} + \left(\frac{q}{p'}\right)^{\frac{1}{p'}} \left(\int_1^\infty \frac{1}{t} \alpha(t)^q \, dt\right)^{\frac{1}{q}} < \infty ,$$

then

$$\| Ax \|_q \leqslant C \| x \|_p .$$

There is also a simpler theorem, like this with $q = p$ and $r = 1$. It includes Hardy's Inequality, with constant C less than 18% above the best possible.

D. C. RUSSELL: <u>Interpolation problems in approximation theory</u>

(Jointly with A.Jakimovski)

Let m be a fixed positive integer and $x = (x_k)_{k \in Z}$ a fixed strictly increasing real sequence, unbounded at both ends. Suppose that Γ is a semi-normed linear space of functions $\mathbb{R} \longrightarrow \mathbb{C}$, and

$$\Gamma^m := \left\{ f \mid f^{(m-1)} \in AC(\mathbb{R}) , f^{(m)} \in \Gamma \right\}, \quad \| f \|_{\Gamma^m} := \| f^{(m)} \|_{\Gamma} .$$

For a real- or complex-valued sequence $y = (y_k)_{k \in Z}$, we define

$$f \in IP(y; \Gamma^m, x) :\Longleftrightarrow f \in \Gamma^m \text{ and } f(x_k) = y_k , \ k \in Z .$$

A unified functional analytic proof gives necessary and sufficient conditions in order that (i) $IP(y; \Gamma^m, x)$ should have a solution; (ii) it should have an <u>optimal</u> (extremal) solution, namely a solution f_* with smallest $\| f_* \|_{\Gamma^m}$. The main requirements are the existence of a Banach function space Λ such that $\Lambda^* \cong \Gamma$ through a Riesz-type Representation Theorem, and the existence of a solid

AK-BK sequence space μ such that $\mathcal{S}_{m,x} \cap \Lambda \cong \mu$, where $\mathcal{S}_{m,x}$ is the space of spline functions of degree m-1 with knots at x. The last requirement involves proving inequalities of the form

$$A \| S \|_{\Lambda} \leq \| \alpha \|_{\mu} \leq A' \| S \|_{\Lambda} \quad \text{for} \quad \alpha \in \mu \ , \ S \in \mathcal{S}_{m,x} \cap \Lambda \ .$$

J. SCHRÖDER: Shape-invariant bounds for vector-valued elliptic-parabolic problems

Let $\Omega \subset \mathbb{R}^m$ be a bounded domain, $u: \bar{\Omega} \longrightarrow \mathbb{R}^n$, G a closed star-shaped set in $\mathbb{R}^n$, $\psi : \bar{\Omega} \longrightarrow \mathbb{R}$,

$$\mathbb{R}^n \ni M\,u(x) = \begin{cases} \mathcal{L}[u](x) + f(x, u(x), u'(x)) & \text{for } x \in \Omega \ , \\ u(x) \quad \text{for } x \in \partial\Omega & , \end{cases}$$

$(\mathcal{L}[u])_i = L[u_i] = -A_i \cdot u_i''$, $A_i(x)$ symmetric and positive definite. We derive differential inequalities for ψ which imply that $u(x) \in \psi(x)\,G$ $(x \in \bar{\Omega})$ for each solution u of $M\,u(x) = r(x)$.

Generalizations: M quasilinear; other boundary terms; estimates $u(x) \in \psi_j(x)\,G_j$ $(j = 1,\dots,N)$.

Examples: (a) $\| u(x) \| \leq \psi(x)$, (b) two sided bounds.

Problems: 1) may $f_i(x,u,u')$ depend on u_k' $(k \neq i)$?
(YES in case (a), NO in case (b)).

2) Is $A_i \neq A_k$ $(i \neq k)$ possible ?
(NO in case (a), YES in case (b)). Intermediate results for other cases.

C. BANDLE: Comparison theorems for second and fourth order elliptic differential equations

Bounds for the solutions of the Dirichlet problem $\Delta u + \alpha u + f(x) = 0$ in $D \subset \mathbb{R}^N$, $u = 0$ on ∂D are constructed by means of symmetrization methods. The estimates are sharp and equality holds for certain radially symmetric problems. Then the results are extended to $\Delta^2 u + a\,\Delta u + b\,u = 1$ in D, $u = \Delta u$ on ∂D. The basic idea is to write this equation as a system of two second order equations and to apply the previous results.

M. C. READE: Subordination and differential equations

The basic idea in this exposition can be expressed best by the following two examples. (1) Let A,B and C be real constants, with $B \geq A \geq 0$ and $C \geq 0$, and let $H(z) = 1 + h_1 z + \dots$ be analytic in the unit disc Δ. If $p(z) = 1 + p_1 z + \dots$ is analytic in Δ, if $p(z)$ satisfies the (Euler) equation

$$A z^2 p''(z) + B z p'(z) + C p(z) = H(z)$$

in Δ, and if $|H(z)| \leq M$ in Δ, then $|p(z)| \leq M$ holds in Δ.

(2) Let s and t be constants, $s > 0$, $\operatorname{Re} t \geq 0$, and let $P(z) = 1 + P_1 z + \dots$ be analytic in the unit disc Δ. If $Q(z) = 1 + Q_1 z + \dots$ is analytic in Δ and if $Q(z)$ satisfies the (Briot-Bouquet) differential equation

$$Q(z) + \frac{z\, Q'(z)}{s\, Q(z) + t} = P(z)$$

in Δ, and if $\operatorname{Re} P(z) > 0$ holds in Δ, then $\operatorname{Re} Q(z) > 0$ holds in Δ.

These results illustrate the notion of (Littlewood) subordination that is at the heart of the results.

These results, and others, are due to P. Eenigenburg, S. Miller, P. Mocanu and the present author.

W. WALTER: A comparison theorem for difference inequalities

In the study of growth and decay properties of nonlinear evolution equations (in particular hyperbolic and parabolic partial differential equations) as $t \to \infty$ the following difference inequality arises (which is satisfied by an energy expression)

$$(*) \qquad \sup_{t \leq s \leq t+1} u(s)^{1+\alpha} \leq C\,(1+t)^r \left(u(t) - u(t+1)\right) + g(t)\,.$$

Nakao and others have derived theorems about the asymptotic behaviour of solutions to (*) as $t \to \infty$, using rather complicated method.

The basis of our treatment is the following theorem (which has a one-line-proof):

Theorem. Let (u_n) and (v_n) be real sequences, $\delta u_n = u_{n+1} - u_n$ and $\delta u_n \leq f_n(u_{n+1})$, $\delta v_n \geq f_n(v_{n+1})$ for $n \geq 0$, where f_n is increasing. Then $u_0 \leq v_0$ implies $u_n \leq v_n$ for all $n \geq 1$.

All the results about (*) which are found in the literature are simple consequences of this theorem. Above that, new results can be derived which are relevant to evolution equations. Three examples: Let $\delta u(t) = u(t+1) - u(t)$ and

$$C\,(1+t)^r\,\delta u(t) \leq g(t) - u(t+1)^{1+\alpha} \quad \text{for } t \geq 0\ ,$$

where $C > 0$, $g \geq 0$, $u \geq 0$. Then for $t \to \infty$ we have

$$\alpha = 0,\ r < 1,\ m \in \mathbb{R}\ : \qquad g(t) = O(t^m) \Longrightarrow u(t) = O(t^m)$$

$$\alpha = 0,\ r = 0,\ m > \log \frac{C}{C+1}\ : \qquad g(t) = O(e^{mt}) \Longrightarrow u(t) = O(e^{mt})$$

$$\alpha > 0,\ m > 0\ : \qquad g(t) = O(e^{(1+\alpha)mt}) \Longrightarrow u\ t\ = O(e^{mt})\ .$$

W. SCHEMPP: <u>Ungleichungen und Symmetrisierung</u>

Das Mitteln über endliche Gruppen oder allgemeiner über kompakte topologische Gruppen und kompakte homogene Mannigfaltigkeiten hat sich als wirksame Methode erwiesen, um z.T. sehr weitreichende Identitäten und Ungleichungen herzuleiten. Ziel des Vortrages ist es, die Wurzeln einiger klassischer Ungleichungen der Approximationstheorie bis auf den Satz von Maschke über die Vollreduzibilität der Darstellungen endlicher Gruppen zurückzuverfolgen und daran anschließend einige neuere Ergebnisse der Kombinatorik über sphärische Pläne darzulegen, die ebenfalls mit Hilfe der Mittelungstechnik ("Symmetrisierung") aus der Ungleichung von Sidelnikov gewonnen werden können.

G. CROSS: <u>On functions with non-negative divided differences</u>

Let $V_n(F) = V_n(F;\ x_0, x_1, \ldots, x_n)$ be the n-th divided difference of F with respect to the n+1 points $x_0, x_1, \ldots, x_n$ on an interval $[a,b]$.

If the inequality $V_n(F) \geq 0$ holds for all choices of points $x_0, x_1, \ldots, x_n$ in $[a,b]$ then F is said to be n-convex on $[a,b]$.

It is shown that if $F(x)$ is n-convex on $[a,b]$, if $F^{(r)}(x)$ exists and is continuous on $[a,b]$, $0 \leq r \leq n-2$, and if $F_{(n-1),+}(a)$ is finite, then

$$F(x) = G(x) + \sum_{k=0}^{n-1} F_{(k),+}(a) \frac{(x-a)^k}{k!} \quad , \quad x \in [a,b] \quad ,$$

where $G(x)$ is monotonic increasing on $[a,b]$, and $F_{(k),+}(a)$ is the one-sided Peano derivative of F at a.

The result has applications to approximation theory.

T. F. BANCHOFF and E. F. BECKENBACH: Counterspherical and countercircular representations

In 1825, in introducing the concept of spherical image for studying the geometry of a surface in 3-space, Karl Friedrich Gauss gave a preliminary treatment of the circular image of a smooth plane curve.

We now introduce and study the countercircular image of a plane curve with respect to a given point in the plane, to complement the second-named author's recently defined counterspherical image of a surface with respect to a given point in space. A principal tool in the study is the notion of the osculating tube of the curve, a concept introduced by the first-named author in collaboration with James H. White.

For space curves, counterspherical images suggested by the Frenet-Serret tangent-normal-binormal apparatus are also briefly investigated.

A. KOVAČEC: Eine algoritmische Methode zum Nachweis von Ungleichungen

Zum Nachweis einer Reihe klassischer Ungleichungen wird über die folgende Idee berichtet: Mittels eines geeignet zu wählenden Funktionals $\tau : \mathbb{R}^n \to \mathbb{R}$ lassen sich die in Rede stehenden Ungleichungen in der Form $\tau(\underline{u}) \leq \tau(\underline{v})$ mit $\underline{v} = \underline{v}(\underline{u})$ schreiben. Man sucht nun eine Abbildung $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ welche den folgenden Bedingungen genügt:

(i) $\tau(\underline{x}) \leq \tau(\Phi(\underline{x}))$, $\underline{x} \in \mathbb{R}^n$

(ii) $\lim_{n\to\infty} \Phi^n(\underline{u}) = \underline{v}$.

Aus (i) und (ii) folgt die erwünschte Ungleichung sofort vermöge

$$\tau(\underline{u}) \leq \tau(\Phi(\underline{u})) \leq \tau(\Phi^2(\underline{u})) \leq \ldots \leq \tau(\Phi^n(\underline{u})) \leq \ldots \leq \tau(\underline{v}).$$

Ein allgemeiner Satz, der aus diesen Überlegungen folgt wird zitiert, und an Anwendungsbeispielen illustriert.

Abwandlungen dieses Konzepts gestatten mittels einer übersichtlichen Matrizenschreibweise die Erzeugung von algebraischen Ungleichungen.

D. BRYDAK: Nonlinear differential inequalities

Let us consider the inequality

$$(1) \qquad \varphi^{(n)}(x) \geqslant g(x, \varphi', \ldots, \varphi^{(n-1)}) .$$

Let us assume that the equation

$$(2) \qquad y^{(n)} = g(x,y,y',\ldots,y^{(n-1)})$$

has n-parameter family F of solutions.

We denote by $R(x,y,y_1,\ldots,y_{n-1})$ the first integral of (2) such that R is increasing with respect to the last variable.

Theorem 1. The function φ is a solution of (1) iff the function

$$\eta(x) := R(x, \varphi(x), \varphi'(x), \ldots, \varphi^{(n-1)}(x))$$

is increasing.

Using the first integral method we can prove a generalization of Polya's generalization of Rolle theorem. Namely, we have

Theorem 2. Let φ be a function which is n times differentiable in $[a,b]$ and let $\varphi(x_i) = y(x_i)$ for $i = 1,2,\ldots,n+1$, where y is a solution of (2) in $[a,b]$, $x_1 < x_2 < \ldots < x_{n+1}$, $x_i \in [a,b]$ for $i = 1,2,\ldots,n+1$. Then there exists a point $\xi \in [x_1,x_{n+1}]$ such that φ satisfies (2) at ξ.

B. CHOCZEWSKI: Differentiable solutions of a functional inequality with two unknown functions (by Z. Powązka)

A paper by Z. Powązka from Kraków is presented.

The inequality is of the form

$$(1) \qquad \varphi_1(G(x,y)) \leqslant F(\varphi_1(x), \varphi_2(y)) , \quad x,y \in I$$

where $F: J^2 \longrightarrow J$, $G: I^2 \longrightarrow I$ are given functions, and $\varphi_i : I \longrightarrow J$ $i = 1,2$, denote unknown functions. Here I and J are some intervals of reals. Differentiable solutions of (1) in some classes of functions are determined.

B. SAFFARI: Trigonometric polynomials and cross-means

Let $f\colon [0,1] \longrightarrow \mathbb{R}$ be a bounded measurable function with $\int_0^1 f(x)dx = 0$ and $f(x) \not\equiv 0$ almost everywhere, so that the numbers $H := \operatorname{ess\,sup} f$ and $h := -\operatorname{ess\,inf} f$ are positive. For $p \in \mathbb{R}$, let

$$M_p(H,h) = \begin{cases} \left(\dfrac{H^p + h^p}{2}\right)^{\frac{1}{p}} & \text{if } p \neq 0 \\ \sqrt{H \cdot h} & \text{if } p = 0 \end{cases}.$$

Also define the "cross-mean" of order p by

$$M_p^*(H,h) = \begin{cases} \left(\dfrac{h\,H^p + H\,h^p}{H + h}\right)^{\frac{1}{p}} & \text{if } p \neq 0 \\ H^{\frac{h}{H+h}} \cdot h^{\frac{H}{H+h}} & \text{if } p = 0 \end{cases}.$$

When $p > 0$, write $\| f \|_p = \left(\int_0^1 |f(x)|^p\,dx\right)^{1/p}$. While studying real trigonometric polynomials

$$\sum_{-n}^{n} c_k \exp 2i\pi kx$$

I observed that $\| f \|_1 \leqslant 2Hh/(H+h) = M_{-1}(H,h)$ and $\| f \|_2 \leqslant \sqrt{Hh} = M_0(H,h)$, with more precise statements for trigonometric polynomials. Whereupon J. Steinig conjectured that $\| f \|_p \leqslant M_{p-2}(H,h)$ whenever $p \geqslant 1$. I disproved this for $1 < p < 2$, but proved it for $p \geqslant 2$ as follows: One easily observes that, for $p \geqslant 1$, $\| f \|_p \leqslant M_p^*(H,h)$, and on the other hand there are several ways of proving that, whenever $p \in \mathbb{R}$,

$$M_p^*(H,h) \begin{cases} \geqslant M_{p-2}(H,h) & \text{if } 1 \leqslant p \leqslant 2 \\ \leqslant M_{p-2}(H,h) & \text{otherwise} \end{cases}.$$

For trigonometric polynomials one obtains, for some values of p, improvements depending on the degree.

Gefördert durch DFG Deutsche Forschungsgemeinschaft

Problems and Remarks

1. Remark (on an inequality for the L^p-norm related to uniform convexity)

Definition. A normed linear space $(\mathcal{L}, \|\cdot\|)$ is uniformly convex when, for any (f_n), (g_n) in $\mathcal{L}$, we have

$$\left(\|f_n\| \leq 1,\ \|g_n\| \leq 1,\ \|\tfrac{1}{2}(f_n+g_n)\| \longrightarrow 1\right) \Longrightarrow \|f_n - g_n\| \longrightarrow 0 .$$

Clarkson proved in 1936 that L^p $(:= L^p(\Omega)$, where Ω is a measurable subset of $\mathbb{R}$) is uniformly convex for $1 < p < \infty$. A proof can also be found in Köthe (Topological Vector Spaces I).

For $p \geqslant 2$, uniform convexity comes at once from:

If $\|\cdot\|$ is the L^p-norm, $p \geqslant 2$, then for complex valued $f,g \in L^p(\Omega)$:

$$(1) \qquad \|f-g\|^p \leqslant 2^{p-1}\left\{\|f\|^p + \|g\|^p - \|f+g\|^p\right\}.$$

For $1 < p < 2$ there is no such simple inequality, and the method of proving uniform convexity is quite involved.

We furnish here an analogue of (1) for the range $1 < p < 2$, which again leads immediately to the uniform convexity of L^p in this case.

Theorem. Let $1 < p \leq 2$, $0 < \lambda < 1$, Ω a measurable subset of $\mathbb{R}$. Then there exists positive constant $C_{p,\lambda}$ such that, for every $f,g \in L^p(\Omega)$,

$$(2) \qquad \|f-g\|^p \leqslant C_{p,\lambda}\left\{(1-\lambda)\|f\|^p + \lambda\|g\|^p - \|(1-\lambda)f + \lambda g\|^p\right\}^{\frac{1}{2}p} \cdot \left\{\int_\Omega \max\left(|f|^p, |g|^p\right)\right\}^{1-\frac{1}{2}p} .$$

D. C. Russell (with A. Jakimovski)

2. Problem (on a functional equation)

Must every $f: \mathcal{A} \longrightarrow \mathbb{C}$ be continuous if

1^o $\mathcal{A}$ is a complex Banach algebra,

2^o $f(ab) = h(b)\,f(a) + g(a)\,f(b)$ for all $a,b \in \mathcal{A}$,

3^o g,h are distinct, nonzero homomorphisms from $\mathcal{A}$ into $\mathbb{C}$?

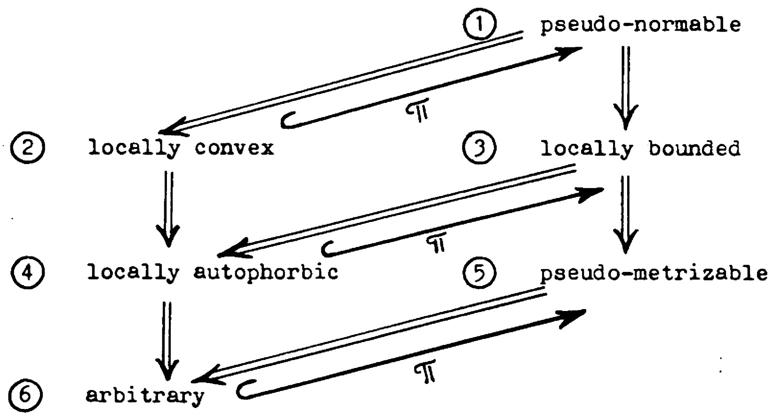
Reference

[1] A. Wilansky: Subspaces, subalgebras and ideals of codimension one in complex algebras, J. London Math. Soc. 9 (1974), 87-92.

A. Wilansky

3. Remark (on locally autophorbic topological vector spaces)

Let $(K, |\cdot|)$ be a topological subfield of $\mathbb{C}$. A subset B of a K-vector space is called <u>autophorbic</u> if for all $\varrho_1, \varrho_2 \in K \cap \mathbb{R}_+$ there exists an $s \in \mathbb{N}$ such that $\varrho_1 B + \varrho_2 B \subset sB$. A topological K-vector space is called <u>locally autophorbic</u> if $\mathcal{O}$ has a neighbourhood base consisting of autophorbic and circled sets. For a K-TVS X the following holds:



None of the converses of these implications holds.

For $r, s \in \{1, \ldots, 6\}$, $(r) \overset{}{\underset{\pi}{\hookrightarrow}} (s)$ means:

(r) holds for X iff X is embeddable into a product of TVS's with property (s).

Acknowledgement: (5) $\not\Rightarrow$ (4) was proved and (4) $\underset{\pi}{\hookrightarrow}$ (3) was suggested by Professor V. L. Klee.

Reference

[1] J. Rätz: On approximately additive mappings, General Inequalities, Vol. 2 (1980), 233-251, especially pp. 245 ff.

J. Rätz

4. Remark (on Prof. Agarwal's lecture)

With respect to the last part of the lecture which deals with the approximate solution of the boundary value problem in question, the following inequality concerning contraction mappings in any metric space M can be used:

If $T: M \longrightarrow M$ satisfies $\varrho(Tx,Ty) \leqslant \alpha \varrho(x,y)$, where $0 \leqslant \alpha < 1$, then

$$\varrho(x,y) \leqslant \varrho(x,Tx) + \varrho(Tx,Ty) + \varrho(Ty,y) .$$

If the middle term on the right is estimated according to the previous inequality and the resulting term is brought to the left, then inequality

$$(*) \qquad \varrho(x,y) \leqslant \frac{1}{1-\alpha}\left\{\varrho(x,Tx) + \varrho(y,Ty)\right\}$$

follows. This basic inequality can be used in many ways (for example, it yields a simple proof of the contraction principle which does not require the summation of geometric series). In the case considered here, x is the fixed point ($x = Tx$) , y is the fixed point of a map T^* ($y = T^*y$) , and $\varrho(Ty,T^*y) \leqslant \varepsilon$. Then inequality (*) gives $\varrho(x,y) \leqslant \frac{\varepsilon}{1-\alpha}$.

W. Walter

5. Problem. There are several infinitesimal characterizations of convex functions. For example, a continuous function f is convex iff

$$\bar{D}^2 f(t) := \limsup_{h \to 0} \left[f(t+h) + f(t-h) - 2f(t)\right] / h^2$$

is nonnegative. Question: Are there similar infinitesimal characterizations (of the if and only if type) for Schur-convex functions ?

W. Walter

6. Remark (on another application of majorization)

We "know" (intuitively ?) that the area (or perimeter) of a (convex) polygon with n+1 sides, inscribed to a circle, is greater than that of a polygon with n sides. This, of course, is not true in general. It clearly is true for regular polygons (may be that is all what we "knew" in the first place).

More generally: Rearrange the (first) polygon so that the length of its sides decrease. (This can be done by rearranging the corresponding central segments, that is, the triangles with the centre of the circle and the two endpoints of the respective sides as vertices.) If, starting from the same point, the vertices of the first polygon precede the corresponding vertices of the (possibly rearranged) second polygon (till they are exhausted), then the first polygon has greater area (and perimeter) than the second.

Proof: The functions $x \mapsto \sin x$ and $x \mapsto \sin 2x$ are concave on $[0, \frac{\pi}{2}]$. So their sums are Schur-concave. ■

Similar statements (with "smaller" instead of "greater") hold for circumscribed polygons with the same proof and, if the sides are small enough to begin with, even for the sum of areas of the inscribed and of the corresponding circumscribed polygons (now again with "greater"). The latter statement is contained in [1] , the former ones are in [3] and "almost" in [2] . For sums of perimeters of inscribed and corresponding circumscribed polygons again the statement with "smaller" holds (not only for small angles).

References

[1] J. Aczél and L. Fuchs: A minimum-problem on areas of inscribed and circumscribed polygons of a circle, Compositio Math. 8 (1950), 61-67.

[2] E. Egerváry: A remark on the length of the circle and on the exponential function, Acta Sci. Math. Szeged 11 (1946), 114-118.

[3] A. W. Marshall and I. Olkin: Inequalities: Theory of majorization and its applications, Academic Press, New York, London, Toronto, Sydney, San Francisco 1979, Ch. 8 E.

J. Aczél

7. Remark (answer to Prof. Wilansky's question)

In connection with W. Walter's Remark 4, A. Wilansky asked whether a transformation $T: M \to M$ ((M,ϱ) - a metric space) satisfying the condition

$$(*) \qquad \varrho(x,y) \leq \frac{1}{1-\alpha}\left\{\varrho(x,Tx) + \varrho(y,Ty)\right\}, \qquad x,y \in M,$$

where $\alpha \in [0,1)$, had to be a contraction.

The answer is: no. For, take $M = \mathbb{R}$, $\varrho(x,y) = |x - y|$, $x,y \in \mathbb{R}$ and $Tx := x + 1 + (1-\alpha)x$, $x \in \mathbb{R}$. T is not a contraction, because otherwise it would have a fixpoint and it has not. On the other hand

$$\frac{1}{1-\alpha}\left(|x - Tx| + |y - Ty|\right) = \frac{1}{1-\alpha}\left(1 + (1-\alpha)|x| + 1 + (1-\alpha)|y|\right) \geqslant$$

$$\geqslant |x| + |y| \geqslant |x - y| , \quad x,y \in \mathbb{R} ,$$

i.e. (*) is satisfied.

R. Ger

8. Remark. In connection with Prof. Ger's talk, in which he reminded the problem of stability of the Cauchy's functional equation

$$f(x+y) = f(x) + f(y) \quad \text{in } \mathbb{R}$$

as it has been formulated by S. Ulam and then solved by D. H. Hyers in 1941, a recent work by E. Turdza from Kraków is to be remarked.

The result reads that the functional equations

$$g\left(\frac{x+y}{2}\right) = \frac{1}{2}\left(g(x) + g(y)\right) \quad \text{and} \quad h(xy) = h(x) + h(y)$$

are both stable in Ulam's sense, the last in $\mathbb{R}$ as well as in $\mathbb{R}\setminus\{0\}$. These cases are not covered by recent works by John Baker and coauthors (cf., in particular, J. Baker, J. Lawrence, F. Zorzitto, The stability of the equation $f(x+y) = f(x)\,f(y)$, Proc. Amer. Math. Soc. 74 (1979) , 242-246).

B. Choczewski

9. Remark. Theorem. Let $u(x,y) \in C^{(2)}([0,a]\times[0,b])$, $u(x,0) = u(0,y) = 0$ for all $0 \leqslant x \leqslant a$, $0 \leqslant x \leqslant b$. Then

$$(1) \quad \int_0^a \int_0^b |u(x,y)\, u_{xy}(x,y)|\,dx\,dy \leqslant C \int_0^a \int_0^b |u_{xy}(x,y)|^2\,dx\,dy ,$$

where

$$C = \frac{ab}{2\sqrt{2}} .$$

Proof: Since,

$$u(x,y) = \int_0^x \int_0^y u_{st}(s,t)\,ds\,dt$$

then, on using the Schwartz's inequality

$$|u(x,y)\, u_{xy}(x,y)| \leqslant |u_{xy}(x,y)| \left(\int_0^x \int_0^y |u_{st}(s,t)|^2\,ds\,dt\right)^{1/2} \sqrt{xy} .$$

Integrating the above and applying the Schwartz's inequality again, we obtain

$$\int_0^a\int_0^b |u(x,y)\, u_{xy}(x,y)|\, dx\, dy \leqslant \left(\int_0^a\int_0^b xy\, dx\, dy\right)^{1/2} \cdot$$

$$\cdot\left(\int_0^a\int_0^b |u_{xy}(x,y)|^2 \left(\int_0^x\int_0^y |u_{st}(s,t)|^2\, ds\, dt\right) dx\, dy\right)^{1/2} \leqslant$$

$$\leqslant \frac{ab}{2}\left(\frac{1}{2}\int_0^x\int_0^y |u_{xy}(x,y)|^2\, dx\, dy\right)^{1/2} ,$$

which is same as (1) .

R. P. Agarwal

10. Problems. (i) Can the constant C in (1) (see Remark 9 above) be replaced by $\frac{ab}{4}$?

(ii) Find the best possible constant for which (1) (see Remark 9 above) holds.

R. P. Agarwal

11. Remark (answer to R. P. Agarwal's question (i))

The choice $u = xy + \varepsilon\Phi$ shows that the constant C in Agarwal's inequality must exceed $\frac{ab}{4}$. Here we assume only that Φ satisfies the boundary conditions, and

$$\int_0^1\int_0^1 \Phi_{xy}\, dx\, dy = 0 , \qquad \int_0^1\int_0^1 (xy\Phi_{xy} + \Phi)\, dx\, dy > 0 .$$

Such a Φ is readily constructed, and $\varepsilon \longrightarrow 0+$ gives a contradiction.

With Wolfgang Walter it was found that the choice $\Phi(x,y) = h(x)\, h(y)$, $h(0) = h(1) = 0$ gives

$$C \geqslant \frac{3+\sqrt{13}}{24}\, ab ,$$

when ε is optimized.

R. Redheffer

12. Problem. Assume that (M,ϱ) is a metric space, $\alpha \in [0,1)$ and T is a transformation of M into itself. If $\varrho(Tx,Ty) \leqslant \alpha\varrho(x,y)$, $x,y \in M$, then, as R. Redheffer said,

$$\varrho(Tx,Ty) \leqslant \alpha\varrho(x,y) \leqslant \frac{\alpha}{1-\alpha}\left(\varrho(x,Tx) + \varrho(y,Ty)\right), \quad x,y \in M$$

(see also Remarks 4 and 7). The question is: does

$$\varrho(Tx,Ty) \leq \frac{\alpha}{1-\alpha}\big(\varrho(x,Tx) + \varrho(y,Ty)\big)$$

imply

$$\varrho(x,y) \leq \frac{1}{1-\alpha}\big(\varrho(x,Tx) + \varrho(y,Ty)\big) \ ?$$

A. Wilansky

13. Remark. This is a short proof for the following

Theorem. Let $T: X \to X$ be a contraction on a compact metric space (X,d). Then T has a fixed point x_o and, for every $x \in X$, $\lim\limits_{n\to\infty} T^n x = x_o$.

Proof: Consider $\tau: X \to \mathbb{R}^+$ defined by $\tau(x) := d(x,Tx)$. τ is continuous, hence, since X is compact, it has a point of minimum, say at x_o.

The assumption $\tau(x_o) > 0$ leads to a contradiction:

$$\tau(x_o) = d(x_o,Tx_o) > d(Tx_o,T^2x_o) = \tau(Tx_o) \quad (Tx_o \in X\ !) .$$

Thus $Tx_o = x_o$. With this x_o define $\hat{\tau}: X \to \mathbb{R}^+$ by $\hat{\tau}(x) := d(x_o,Tx)$. Since X is compact there exists a sequence $(n_k)_{k\in\mathbb{N}}$ such that $p := \lim\limits_{k\to\infty} T^{n_k}x$ exists. Assume $\hat{\tau}(p) > 0$; this leads to the following contradiction

$$\alpha := \lim_{n\to\infty} \hat{\tau}(T^n x) = \lim_{k\to\infty} \hat{\tau}(T^{n_k}x) = \hat{\tau}(p) > \hat{\tau}(Tp) = \hat{\tau}\big(T \lim_{k\to\infty} T^{n_k}x\big)$$

$$= \lim_{k\to\infty} \hat{\tau}(T^{n_k+1}x) = \lim_{n\to\infty} \hat{\tau}(T^n x) = \alpha .$$

Thus $\hat{\tau}(p) = 0$, which means $\lim\limits_{n\to\infty} \hat{\tau}(T^n x) = 0$, i.e. $x_o = \lim\limits_{n\to\infty} T^n x$.

(The limits mentioned exist since the corresponding sequences are monotone decreasing and bounded by 0).

A. Kovačec

14. Problems. 1^o Let $a,p \in R$, $x = (x_1,\ldots,x_n) \in R^n_+$ and

$$M_{a,p}(x) = \begin{cases} \left(\dfrac{\sum_{i=1}^{n} x_i^{a+p}}{\sum_{i=1}^{n} x_i^{p}} \right)^{\frac{1}{a}} & \text{if } a \neq 0 \\ \exp \dfrac{\sum_{i=1}^{n} x_i^{p} \ln x_i}{\sum_{i=1}^{n} x_i^{p}} & \text{if } a = 0 . \end{cases}$$

Find necessary and sufficient conditions for $(a,p,b,q,c,r) \in R^6$ such that the inequality

(1) $$M_{a,p}(x \cdot y) \leq M_{b,q}(x)\, M_{c,r}(y)$$

holds for all $x,y \in I^n$, $n = 1,2,\ldots$ where $x \cdot y = (x_1 y_1, \ldots, x_n y_n)$, I is a fixed open subinterval of $R_+ = (0,\infty)$. If $I = R_+$ the necessary and sufficient conditions are known (see Zs. Páles, On Hölder type inequalities, to appear in J. Math. Anal. Appl.).

(1) is satisfied for $x,y \in I^n$ if and only if

(2) $$j_{a,p}(u \cdot v) \leq j_{b,q}(u) + j_{c,r}(v) \quad \text{for } u,v \in \left(\frac{m}{M}, \frac{M}{m}\right)$$

(see L. Losonczi, Subadditive Mittelwerte, Archiv der Math. 22 (1971), 168-174).

2^0 Suppose now that (1) holds for all $x,y \in R_+^n$, $n = 1,2,\ldots$ and let I_1 be a compact subinterval of R_+. Find the best constant C such that the inequality

(3) $$C \leq \frac{M_{a,p}(x \cdot y)}{M_{b,q}(x)\, M_{c,r}(y)}$$

holds for all $x,y \in I_1^n$, $n = 1,2,\ldots$. Several results are known if $p = q = r = 0$ (see D. S. Mitrinović, Analytic inequalities, Springer-Verlag, Berlin-Heildelberg-New York 1970).

3^0 Let φ, ψ be differentiable functions on R_+ with positive derivative and f,g be positive functions on R_+. Let further

$$M_{\varphi,f}(x) = \varphi^{-1}\left(\frac{\sum_{i=1}^{n} f(x_i)\, \varphi(x_i)}{\sum_{i=1}^{n} f(x_i)} \right), \quad x = (x_1,\ldots,x_n) \in R_+^n$$

and $M_{\psi,g}$ be defined similarly.

Does the inequality

$$(4) \qquad \frac{1}{n}\sum_{i=1}^{n} x_i y_i \leqslant M_{\varphi,f}(x)\, M_{\psi,g}(y) , \qquad x,y \in R_+^n$$

imply the existence of a constant $p > 1$, such that

$$(5) \qquad \frac{1}{n}\sum_{i=1}^{n} x_i y_i \leqslant \left(\frac{1}{n}\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \left(\frac{1}{n}\sum_{i=1}^{n} y_i^q\right)^{\frac{1}{q}} \leqslant M_{\varphi,f}(x)\, M_{\psi,g}(y)$$

holds, where $\frac{1}{p} + \frac{1}{q} = 1$. If $f(u) = g(u) = 1$ and φ , ψ are twice differentiable then (4) implies (5) . If in (4) $M_{\varphi,f}$, $M_{\psi,g}$ are replaced by deviation means then in general (4) does not imply (5) (see my talk: Remarks on Hölder inequality) .

L. Losonczi

15. Remark. Consider the system consisting of the inequality

$$\psi^n(x) \leqslant g(x)$$

(with the n-th iterate of ψ) and of the equation

$$\psi(g(x)) = g(\psi(x))$$

of commuting functions. Here g is a self-mapping of an interval $[0,a]$, continuous and strictly increasing, $0 < g(x) < x$ in $(0,a)$, $g(a) = a$ and $g(x) \sim x^p$, $p > 0$, as $x \to 0+$. The problem reads as follows:

Given a continuous solution ψ of the system with the property $\psi(x) \sim x^{p^{1/n}}$ find a continuous n-th iterative root φ of g : $\varphi^n(x) = g(x)$ in $[0,a]$, majorizing ψ and enjoying the same asymptotic property as ψ does.

For n = 2 the problem has been settled by E. Turdza (Comparison theorems for a functional inequality, General Inequalities I, p. 199-211) . Same result has recently been obtained, for an arbitrary n, by M. Czerni from Kraków.

B. Choczewski

16. Problem. Let X be an ordered topological vector space and let A be a linear operator in X. Let us assume that the sequence $A^n x$ is convergent for every $x \in X$. Let K be the set of all solutions of the inequality

$$A x \leqslant x \quad , \quad x \in X .$$

What are conditions for the equality

$$X = K - K$$

to be satisfied ?

D. Brydak

Compiled by R. Ger (Katowice)

List of participants

ACZÉL, J.; Dept. of Pure Math., University of Waterloo, Waterloo, Ont. N2L 3G1, Canada

AGARWAL, R. P.; Matscience, Madras, 600020 India

ALSINA, C.; Dep. Matemáticas i Estadistica, Universitat Politecnica de Barcelona, Barcelona 25, Spain

BANCHOFF, T. F.; Dept. of Math., Brown University, Providence, Rhode Island, 02912 U.S.A.

BANDLE, C.; Math. Inst., Universität Basel, Rheinsprung 21, CH-4051 Basel, Switzerland

BECKENBACH, E. F.; Dept. of Math., University of California, Los Angeles, California 90024 U.S.A.

BRYDAK, D.; Inst. of Math., Wyższa Szkoła Pedagogiczna, Podchorążych 2, Kraków, Poland

CHOCZEWSKI, B.; Inst. of Math., University of Mining and Metallurgy, al. Mickiewicza 30, C-4, 30-059 Kraków, Poland

CLAUSING, A.; Institut für Mathematische Statistik, Einsteinstr. 64, D-4400 Münster, West Germany

CROSS, G.; Dept. of Pure Math., University of Waterloo, Waterloo, Ont. N2L 3G1, Canada

DARÓCZY, Z.; Dept. of Math., L. Kossuth University, H-4010 Debrecen, Hungary

EICHHORN, W.; Institut für Wirtschaftstheorie und OR , Universität Karlsruhe, D-7500 Karlsruhe, West Germany

FEHÉR, F.; Lehrstuhl A für Mathematik, RWTH Aachen, Templergraben 55, D-5100 Aachen, West Germany

FENYÖ, I.; Technical University in Budapest, H-1111 Budapest, Stoczek u. 2, Hungary

GER, R.; Dept. of Math., Silesian University, Bankowa 14, 40-007 Katowice, Poland

GOLDBERG, M.; Dept. of Math., Technion, Haifa, Israel

KOVAČEC, A.; An der Niederhaid 21, A-1140 Wien, Austria

LACZKOVICH, M.; Eötvös Loránd University, Department I. of Analysis, Budapest, Múzeum krt. 6-8, H-1088 Hungary

LOSONCZI, L.; Dept. of Math., L. Kossuth University, H-4010 Debrecen, Hungary

LOVE, E. R.; Dept. of Math., University of Melbourne, Parkville, Victoria 3052, Australia

MARSHALL, A. W.; Dept. of Math., University of British Columbia, Vancouver B.C.,V6T 1Y4 Canada

NESSEL, R. J.; Lehrstuhl A für Mathematik, RWTH Aachen, Templergraben 55, D-5100 Aachen, West Germany

OLKIN, I.; Dept. of Statistics, Stanford University, Stanford, California 94305, U.S.A.

PÁLES, Z.; Dept. of Math., L. Kossuth University, H-4010 Debrecen, Hungary

RÄTZ, J.; Math. Inst., Universität Bern Sidlerstr. 5, CH-3012 Bern, Switzerland

READE, M. O.; Dept. of Math., University of Michigan, Ann Arbor, Michigan 48109, U.S.A.

REDHEFFER, R.; Dept. of Math., University of California, Los Angeles, California 90024, U.S.A.

RUSSELL, D. C.; Dept. of Math., York University, Downsview (Toronto) Ontario, M3J 1P3 Canada

SAFFARI, B.; Mathématiques Université d'Orsay, Batiment 425, 91405 Orsay, France

SCHEMPP, W.; Lehrstuhl für Mathematik I, Universität Siegen, Hölderlinstr. 3, D-5900 Siegen 21, West Germany

SCHNITZER, F. J.; Inst. Math., Montan-Universitat, 8700 Leoben, Austria

SCHRÖDER, J.; Math. Inst., Universität Köln, Albert Magnus-Platz, D-5000 Köln, West Germany

SCHWEIZER, B.; Dept. of Math. and Statistics, University of Massachusetts, Amherst, MA 01003 U.S.A.

STEINIG, J.; Institut Mathématique, Université de Genève, C.P. 124, 1211 Genève 24, Switzerland

VOLKMANN, P.; Math. Inst. I, Universität Karlsruhe, D-7500 Karlsruhe, West Germany

WALTER, W.; Math. Inst. I, Universität Karlsruhe, D-7500 Karlsruhe, West Germany

WANG, CH.; Dept. of Math. and Statistics, University of Regina, Regina, Saskatchewan, S4S 0A2 Canada

WILANSKY, A.; Dept. of Math. # 14, Lehigh University, Bethlehem, PA 18015 U.S.A.

ZELLER, K.; Sonnenstr. 11, D-7400 Tübingen, West Germany

Gefördert durch
DFG Deutsche Forschungsgemeinschaft