MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 24/1983

Special Functions and Differential Equations in the Complex Domain

June 6 - 10, 1983

The conference was directed by Friedrich W. Schäfke (Konstanz) and Dieter Schmidt (Essen). Main field of interest were the Theory of Special Functions of Mathematical Physics (especially of higher Special Functions as Mathiéu-, Spheroidal-, Lamé-, and Heun-Functions), the general theory of singular linear differential equations in the complex domain (here in particular connection problems) and related questions.

Two lectures (H. E. Benzinger, B. L. J. Braaksma) were attended together with the participants of the parallel conference on "Spectral Theory of Ordinary Differential Operators".

Abstracts

HAROLD E. BENZINGER:

Spectral Theory of Ordinary Differential Operators

Differential operators defined by Birkhoff regular boundary conditions generate eigenfunction systems with convergence properties similar to those of Fourier series. The definition of Birkhoff regularity has no intrinsic spectral theoretic content. We show that the resolvent operators of Birkhoff regular operators with simple spectrum are well-bounded operators. This implies the existence of a functional calculus and an integration theory with respect to a family of projections. This can be applied to the study of semigroups generated by even order differential operators.

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D. A. LUTZ

Connection Problems for Differential Equations of Rank Two or More

The lateral connection problem for systems of linear differential equations of the form $x' = (z^{r-1} \Sigma_0^\infty A_\nu z^{-\nu})x$, where A_0 has all distinct eigenvalues, can be transfered to a connection problem for a system of associated functions which are constructed using the formal solutions. For $r = 1$, the associated functions have only regular-type singularities in the finite plane, but for $r \geq 2$ the singularities can be irregular. However, they are convolutions of functions having regular-type singularities with some functions that are LAPLACE transforms of exponential polynomials. In this talk, associated functions were constructed when $r = 2$ and in the simpler case when the singularities are regular. A full discussion for the case $r = 1$ can be found in SIAM J. Math. Anal. 1981 (BALSER, JURKAT, and LUTZ). The cases $r \geq 2$ can be found in J. Math. Anal. Appl. 1982 by the same authors.

W. BALSER

Reihenentwicklungen für die Stokes - Multiplikatoren gewisser Differentialgleichungen

Betrachtet werden die Differentialgleichungen der Form

$$(1) \qquad zx' = (z\Lambda + A_1)x$$

mit einer Diagonalmatrix Λ mit verschiedenen Diagonalelementen. Ihre Stokes'schen Multiplikatoren sind ganze Funktionen in den Außerdiagonalelementen von A_1 . Die Koeffizienten der Potenzreihenentwicklung der Stokes'schen Multiplikatoren ergeben sich aus den Stokes'schen Multiplikatoren gewisser Differentialgleichungen, die analog wie (1) gebaut sind, wobei aber A_1 eine sehr einfache Matrix ist. Für diese "einfachen" Gleichungen wird die Natur der Stokes'schen Multiplikatoren als Funktionen der Elemente von Λ untersucht und ihre Potenzreihenentwicklung (in diesen Variablen) um den Punkt ∞ angegeben.

R. SCHÄFKE

Connection Coefficients for a Regular and an Irregular Singular Point

Consider a system of n linear differential equations of the form

$$(D) \qquad y'(z) = \left(\frac{1}{z} A_o + \frac{1}{(z-1)^2} B + \frac{1}{z-1} A_1 + G(z)\right) y(z)$$

where A_o, A_1, B are n by n matrices, $G(z)$ is matrix valued and analytic in $|z| < r,\ r > 1$. It has a singular point of the first kind at 0, a singular point of the second kind at 1 and no further singular points within $|z| \leq 1$.
At 0, there are Floquet solutions of (D) of the form

$$y_o(z) = z^{\alpha} \sum_{k=o}^{\infty} z^k d_k \quad (|z| \leq 1) \quad \text{with } d_k \in \mathbb{C}^n .$$

We assume that B has distinct eigenvalues. Then there exists a unique fundamental system $y_1(z),\dots,y_n(z)$ of solutions having simple asymptotic behaviour as $z \to 1$ in $-\frac{\pi}{2} < \arg(1-z) < \frac{\pi}{2} + \varepsilon$.
In my talk I presented a method for the computation of the connection coefficients γ_j in

$$y_o(z) = \sum_{j=1}^{n} \gamma_j\, y_j(z)$$

via the power series for $y_o(z)$ from the data of (D).

M. KOHNO

Connection Problems for Logarithmic Solutions

We consider connection problems in a non-generic (logarithmic) case for the hypergeometric system

$$(*) \qquad (t-B)\frac{dX}{dt} = AX$$

and the Birkhoff system

$$(**) \qquad t\frac{dX}{dt} = (A_o + A_1 t + \dots + A_q t^q) X .$$

Then we can prove that for a suitably chosen fundamental matrix solution of (*) there holds the extended Gauß-Kummer's formula. On the other hand, as for (**), we can show the Frobenius-like theorem, i.e., the Stokes multipliers for logarithmic solutions are given by derivatives of those of a non-logarithmic solution with respect to the characteristic exponent.

F. W. SCHÄFKE:

Ein "Riemann-Symbol" für (konfluente) Fuchssche Differentialgleichungen 2. Ordnung

Schreibt man die Differentialgleichung mit Hilfe der n endlichen einfachen Singularitäten, den Indizes dort, mit den Exponenten in ∞ und mit n geeignet (!) eingeführten Konstanten, so ergeben sich ausserordentlich einfache Invarianz- und Transformationseigenschaften. Sie lassen viel Bekanntes und Neues in sehr durchsichtiger Weise erkennen.

A. SEEGER:

Heunsche Differentialgleichung, Heunsche Funktionen und ihre Anwendungen

Die Heunsche Differentialgleichung, auf die jede Differentialgleichung 2. Ordnung der Fuchsschen Klasse mit vier außerwesentlich singulären Stellen reduziert werden kann, ist bis jetzt vor allem im Zusammenhang mit der Separation der Potentialgleichung und der Wellengleichung in krummlinigen Koordinaten näher betrachtet worden, wobei als die am ausführlichsten untersuchten Spezialfälle die Lamésche und die Mathieusche Differentialgleichung sowie die mit der Differentialgleichung der Shäroidwellenfunktionen zusammenhängende spezielle Heunsche Gleichung zu gelten haben. In zunehmendem Maße tauchen bei Anwendungen in Physik und Technik auch allgemeinere Heunsche Gleichungen und neuartige Fragestellungen auf.

Hauptziel des Vortrages ist es, eine Diskussion darüber in Gang zu bringen, inwieweit die vorhandene Theorie der Differentialgleichungen der Fuchsschen Klasse für die bei der Heunschen Differentialgleichung auftretenden Fragestellungen genutzt werden kann und ob ähnlich wie bei der hypergeometrischen Differentialgleichung ein für Anwendungen brauchbarer Überblick über die Lösungen der Heunschen Gleichung und ihren konfluenten Spezialfällen gewonnen werden kann.

ST. GRADEK:

Ein reelles Additionstheorem für Sphäroidfunktionen

Unter einem Additionstheorem für Sphäroidfunktionen versteht man die Entwicklung einer in gestreckt-rotationselliptischen Koordinaten separierten Lösung der Schwingungsgleichung nach Lösungen der Schwingungsgleichung, die in einem anderen Koordinatensystem des gleichen Typs separiert sind. Hier wurde der Fall behandelt, daß des zweite Koordinatensystem durch eine Translationsbewegung aus dem ersten hervorgeht, wobei jedoch verschiedene Exzentrizitäten erlaubt sind.

C. HUNTER:

The Eigenvalues of Mathieu's and the Spheroidal Wave Equation for Complex Values of their Parameters

Mathieu's equation and the spheroidal wave equation, for some fixed angular wavenumber, provide two examples of problems in which the eigenvalue λ depends on some second parameter, which we shall label as q. Using WKBJ (phase integral) methods based on the assumption that $|\lambda|$ is large, approximate relations between λ and q are obtained. These relations appear to be valid uniformly throughout the complex q-plane; they match known expansions for both large and small values of q. They also predict a doubly infinite array of locations in the complex q-plane of the square-root branch points at which some two eigenvalues become equal. As a consequence of these branch points, all the eigenvalues of any one kind can be seen to be values of one many-valued function.

H. VOLKMER:

Integraldarstellungen von Produkten Laméscher Funktionen mit Hilfe von Fundamentallösungen

Gegenstand meines Vortrages sind Integraldarstellungen Laméscher Funktionen, die aus der Theorie der Fundamentallösungen geeigneter partieller Differentialgleichungen gefolgert werden können. Die Kerne dieser Darstellungen enthalten die Legendrefunktion Q_ν. Ich verallgemeinere und verbessere Integraldarstellungen für äußere elliptische harmonische Funktionen, die von Erdélyi stammen und für Lamésche Funktionen der 2. Art, die von Shail studiert wurden.

G. K. IMMINK:

Gevrey Classes in the Theory of Difference Equations

My talk is concerned with n-dimensional systems of linear difference equations of the form: $y(s+1) - A(s)y(s) = f(s)$, where both A and A^{-1} are meromorphic at ∞ and f is holomorphic at ∞.
In my Ph.D. thesis I have proved some existence theorems for holomorphic solutions of such equations with a prescribed asymptotic behaviour in appropriate sectors of the complex plane. In particular I have looked for solutions belonging to certain Gevrey-classes. One of the main results is the following:
Let $H_j(R) = \{s \in \mathbb{C} : -\frac{\pi}{2} + j\pi < \arg s < \frac{\pi}{2} + j\pi;\ |s| \geq R\}$, $R > 0$, $j = 0,1,2,3$. Suppose that neither 0 nor π is a "singular direction" of the (homogeneous) equation:
$y(s+1) - A(s)y(s) = 0$. If R is sufficiently large, then there exist matrix functions F_j, holomorphic in $H_j(R)$ $(j = 0,1,2,3)$ such that: 1. F_j and F_j^{-1} belong to a certain Gevrey class of holomorphic matrix functions with an asymptotic expansion.
2. $F_j(s+1)^{-1} A(s) F_j(s) = A^c(s)$ in $H_j(R)$, where $A^c(s)$ denotes a canonical (or normal) form of $A(s)$.

E. WAGENFÜHRER:

Zur Reduktion der Polordnung eines linearen Differentialgleichungssystems an einer singulären Stelle

We consider the systems of n linear differential equations

$$(*) \quad (Dy)(x) := xy'(x) - A(x)y(x) = 0, \quad A(x) = x^{-s} \sum_{\nu=0}^{\infty} x^{\nu} B_{\nu}, \quad s = s(A) \in \mathbb{N},$$

in a neighbourhood of $x = 0$. We want to determine $\ell(A)$ = minimum of all $s(\tilde{A})$, where the $\tilde{A}$ are generated by meromorphic transformations of (*). For this, we use the quantities ρ_{r+1} $(r = 0,1,\ldots,s)$ introduced by Gérard and Levelt in 1973. The question is, how to determine the ρ_{r+1} practically. For this, we introduce certain matrices $A_m^{[r]}(\lambda,\kappa)$ which are derived from the Laurent series expansion of $A(x)$ and contain two linear complex parameters λ,κ. Let $d^{(r)}$ be the maximum defect number of all $A_m^{[r]}$. Then we have $\rho_{r+1} = n(s-r) - d^{(r)}$ and, consequently, $\ell(A) \leq r$ iff $d^{(r)} = n(s-r)$.

R. SCHÄFKE:

Stability and Identification of Formal Meromorphic Invariants

Consider a differential equation in $\mathbb{C}^n$ having a pole at 0

$$(D) \quad z^{s+1} y'(z) = A(z)y(z) \quad \text{where} \quad A(z) = \sum_{\nu=0}^{\infty} A_{\nu} z^{\nu} \quad (|z| \text{ small}).$$

If $s \in \mathbb{N}$ is not zero, (D) has a singular point of the second kind at 0 and, since no assumptions on $A(z)$ are made, it can be a very complicated singular point. (D) has a formal fundamental solution of the form

$$H(z) = F(z^{1/p})\, z^J \exp(Q(z^{-1/p}))$$

where $p \in \mathbb{N}'$, $Q(z^{-1/p})$ is a diagonal matrix of polynomials in $z^{-1/p}$ (without constant term), J is an n by n matrix commuting with Q and $F(z^{1/p})$ and $F(z^{1/p})^{-1}$ are formal power series in $z^{1/p}$.

The entries of Q are invariant with respect to formal meromorphic transformations. In my talk I presented results of D. Lutz and me partially answering the questions

(1) How stable are the entries of Q with respect to perturbations $A(z) \to A(z) + z^N B(z)$.

(2) Identify $Q(z^{-1/p})$ (and other known invariants) in terms of $A(z)$.

PO-FANG HSIEH:

Global Simplification of Singularly Perturbed Linear Ordinary Differential Equations in the Complex Domain

Recently, Gingold and Hsieh proved a global simplification theorem for a system of equations

(1) $$\varepsilon^h dy/dx = \{A(x) + g(\varepsilon)Q(x,\varepsilon)\}y$$

according to the eigenvalues of $A(x)$, in a lense shape complex domain under some conditions on the analytivity of the coefficients and the domain D. Assume that $A(x)$ is a diagonal matrix. (1) is said to be an almost diagonal system if there exists a matrix $P(x,\varepsilon)$ such that $\lim P = 0$, as $\varepsilon \to 0$, and the transformation $y = (I+P)v$ reduces (1) to $\varepsilon^h dv/dx = A(x)v$. Global existence of such $P(x,\varepsilon)$ will be examined.

Y. SIBUYA:

A Generalization of Phragmén-Lindelöf Theorem

In Phragmén-Lindelöf Theorem, a function is considered in a sector.

We consider the case where a sector is covered by small open sectors, and, in each small sector, a function is given. We explain a theorem concerning this family of functions, similar to Phragmén-Lindelöf Theorem.

R. GERARD:

Formal and Convergent Solutions of Singular Equations

Let $\hat{O} = A[[x_1, x_2, \ldots, x_n]] = A[[x]]$ the ring of formal power series with coefficients in a valued ring A. $\hat{O} = A\{x\}$ the ring of convergent series.

$$P : \hat{O}^q \longrightarrow \hat{O}^q$$

$$u \longrightarrow P_u = \sum_{m=o}^{+\infty} \left(\sum_{\substack{k_o \le k \le m \\ |k| \ge o}}^{+\infty} P_k(m-k)\, u_{m-k} \right) x^m$$

where $R_q \in \mathbb{Z}^n$, $P_k(\ell)$ a matrix valued function of ℓ such that $P(\hat{O}^q) \subset \hat{O}^q$. We give first several examples of such equations including singular partial differential equations.
We then give conditions for P such that $P_u = f$ has an analytic solution and solve nonlinear equations

$$P_u = F(x,u) \ , \quad F \text{ analytic near}$$

the origin of $\mathbb{C}^n \times \mathbb{C}^q$. For applications we treat also the cases with parameters and give a proof of a theorem of S. Kaplan. These results were obtained by G. Bengel and R. Gérard.

F.M. ARSCOTT:

Co-existence of Simply-Periodic Solutions of a Doubly-Periodic Equation

The differential equation of Lamé is

$$\frac{d^2w}{dz^2} + (h-\gamma(\gamma+\ )k^2 \mathrm{sn}^2 z)w = 0$$

whose coefficients are doubly-periodic with periods $2K$ and $2iK'$. It is well known that this equation has solutions (the so-called Lamé polynomials) with $2K$ and $2iK'$ as period or half-period, if and only if $\gamma = n$, an integer, and h is a characteristic value.

There are certain other value-pairs γ, h, however, for which the equation has a solution with $2K$ as period or halfperiod, and an independent solution with $2iK'$ as period or half-period. This talk will describe these eigenvalue-pairs, and properties of the corresponding solutions.

B. D. SLEEMAN:

Doubly-Periodic Floquet Theory

The purpose of the paper is to develop a theory for differential equations with doubly-periodic coefficients analogous to the classic Floquet theory for differential equations with singly periodic coefficients. Unlike the classical theory the role of the exponent ν of the differential equation is fundamental. If ν takes integral values the analogous theory is well-known and goes back to the work of Hermitê in 1811. When ν is rational the theory depends essentially on wether a certain number theoretic conjecture proposed by F. M. Arscott and G. P. Wright (1969) is true.
The paper resolves the conjecture and brings the doubly-periodic Floquet theory to some degree of completion.

B. L. J. BRAAKSMA:

Asymptotics and Deficiency Indices for Certain Pairs of Differential Equations

Let $M = \prod_1^n (xD-a_j) - \mu x^\alpha \prod_1^m (xD-b_j)$, $D = \frac{d}{dx}$, $0 \leq m < n$.

Solutions of $My = 0$ have been studied by Meijer, Kohno and Okkohchi who solve the connection problem in the general case except for certain exceptional cases. We give an outline of the method of Meijer and show how the restriction can be removed.

Next we consider perturbations of M with the property that their solutions have the same behavior as $x \to +\infty$ as those of M . This is used to determine deficiency indices related to $L_1 y = \lambda L_2 y$, where L_1, L_2 are formally symmetric, one of them is positive, and L_1, L_2 are perturbations of operators $x^\gamma M_1$, $x^\delta M_2$ with M_1 and M_2 of the same type as M .

MOURAD E. H. ISMAIL:

Orthogonal Polynomials and Spectra of Jacobi Matrices

I will discuss the relationship between spectra of positive definite Jacobi matrices and orthogonal polynomials. I will present a method to obtain the measure that a set of orthogonal polynomials is orthogonal with respect to from the three term recurrence relation satisfied by the polynomials. This equivalent to computing the spectral measure of the associated Jacobi matrix. Application to differential equations will also be mentioned. Examples will also be discussed.

TH. KURTH/D. SCHMIDT:

The Solution of Second Order Differential Equations with a Rank One Singularity at Infinity by Series in Terms of Confluent Hypergeometric Functions

We consider the differential equation

$$\text{(D)} \qquad y'' + \left(\frac{1}{z} + \frac{a_1(z)}{z^2}\right)y' + \left(1 + \frac{2i\kappa}{z} + \frac{a_2(z)}{z^2}\right)y = 0 \quad ,$$

where $\kappa \in \mathbb{C}$ and a_1, a_2 are holomorphic in a full neighbourhood of infinity $0 \leq r < |z| \leq \infty$. (D) is a normal form of the general linear second order differential equation with a rank one singularity at ∞ . In the special case $a_1(z) \equiv 0$ and $a_2(z) \equiv -\mu^2$ $(\mu \in \mathbb{C})$ (D) essentially becomes the confluent hypergeometric equation.

We show, that the solutions of (D) can be represented by series in terms of confluent hypergeometric functions, which are valid on the whole Riemannian surface of $\ln z$ over $r < |z| < \infty$, and which allow to describe the full analytic behavior of these solutions at the singular point ∞ . As a consequence we also obtain a formula for calculating the Stokes multipliers of (D) .

Berichterstatter: R. Schäfke

Participants

Arscott, F.M. Prof. Dr., Department of Mathematics,
University of Manitoba, Winnipeg, Manitoba, Canada

Balser, W., Priv.-Doz., Dr., Abteilung für Mathematik,
Universität Ulm, Oberer Eselsberg, 7900 Ulm

Braaksma, B. L. J., Prof. Dr., Department of Mathematics,
University of Groningen, P.O. Box 800,
Groningen, Niederlande

Gérard, R., Prof., Dr., Département de Mathématique,
Université Louis Pasteur IRMA, 7 rue René Descartes,
6700 Strasbourg Cédex, France

Gradek, St., Dr., Fachbereich Mathematik, Universität Konstanz,
Postfach 7733, 7750 Konstanz

Hsieh, Po-Fang, Prof. Dr., Department of Mathematics,
Western Michigan University, Kalamazoo,
Michigan 49008, USA

Hunter, C., Prof. Dr., Math. and Computer Science Department,
Florida State University, Tallahassee, Florida 32306

Immink, G. K., Dr., Department of Mathematics, University
of Groningen, P.O. Box 800, Groningen, Niederlande

Ismail, Monrad E.H., Dr., Department of Mathematics, Arizone
State University, Tempe, Arizona 85281, USA

Kohno, M., Prof. Dr., Department of Mathematics, Faculty
of Science, Hiroshima University, Hiroshima, Japan

Kurth, Th., Dr., Fachbereich Mathematik, Universität
Konstanz, Postfach 7733, 7750 Konstanz

Lutz, D. A., Prof.Dr., University of Wisconsin, Milwaukee,
Wisconsin 53201, USA

Sattler, A., Prof. Dr., Fachbereich Mathematik, Universität
Essen - GHS, Universitätsstr. 3, 4300 Essen 1

Seeger, A., Prof. Dr., Institut für Physik, Heisenbergstr. 1,
7000 Stuttgart 80

Sibuya, Y., Prof. Dr., School of Mathematics, University
of Minnesota, Minneapolis, Minnesota 55455, USA

Sleeman, B. D., Prof. Dr., Department of Mathematics, University of Dundee, Dundee, DD1 4HN, Großbritannien

Schäfke, F. W., Prof. Dr., Fachbereich Mathematik, Universität Konstanz, Postfach 7733, 7750 Konstanz

Schäfke, R., Dr., Fachbereich Mathematik, Universität Essen - GHS, Universitätsstr. 3, 4300 Essen 1

Schmidt, D., Prof. Dr., Fachbereich Mathematik, Universität Essen - GHS, Universitätsstr. 3, 4300 Essen 1

Volkmer, H., Dr., Fachbereich Mathematik, Universität Essen - GHS, Universitätsstr. 3, 4300 Essen 1

Wagenführer, E., Dr., Fachbereich Mathematik, Universität Regensburg, Universitätsstr. 3, 8400 Regensburg

Wasow, W., Prof. Dr., University of Wisconsin, Van Vleck Hall, Madison, Wisconsin 53706, USA

Wolf, G., Dr., Fachbereich Mathematik, Universität Essen - GHS, Universitätsstr. 3, 4300 Essen 1